GRAPH THEORY

ANDREW WOOD

1. Preliminaries

1.1. **Basic definitions.** A simple undirected *graph* G consists of a set $V(G)$ of *vertices* together with an *adjacency relation* $\sim_G$ which is an irreflexive symmetric binary relation on $V(G)$.

Let $u, v \in V(G)$. We say that u and v are *adjacent* to if $u \sim_G v$. In this case, we also say that $\{u, v\}$ is an *edge* of G *incident* with both u and v, and we sometimes denote the set of edges by $E(G)$.

A *subgraph* of a graph G is a graph whose vertices all belong to $V(G)$ and whose edges belong to $E(G)$. An *isomorphism* from G_1 to G_2 is a bijection $f : V(G_1) \to V(G_2)$ such that f preserves adjacency - meaning that for all $u, v \in V(G_1)$ we have

$$u \sim_{G_1} v \Longleftrightarrow f(u) \sim_{G_2} f(v).$$

We then say that G_1 and G_2 are *isomorphic* if there is an isomorphism between them.

The number $|V(G)|$ is called the *order* of G. The *degree*, or *valency*, of a vertex v is the number of edges incident with it, and denoted by $\deg(v)$. A vertex adjacent to all other vertices of G is said to be *universal*.

Lemma 1.1. (The Handshaking Lemma). *If G is a simple finite graph, then*

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

Proof. Count in two different ways the number of pairs (v, e) where e is an edge incident with the vertex v. □

A graph in which every vertex has the same degree k is said to be a *k-regular graph*, or a *regular graph* of *degree k*.

If the vertex-set of a graph G is the union of two disjoint subsets V and W such that every edge of G joins a vertex in V to a vertex in W, then G is said to be *bipartite*, with parts (or bi-parts) V and W.

1.2. **Examples.** A simple graph in which every two vertices are adjacent is called a *complete graph*. The symbol K_n is used to denote the complete graph on n vertices.

A graph with no edges is called an *edgeless graph* (or a *null graph*). The edgeless graph on n vertices is often denoted by $\overline{K_n}$ (or sometimes by N_n).

The *circuit graph* (or *cycle graph*) C_n has n vertices $v_0, v_1, \dots, v_{n-1}$, with single edges joining v_i to v_{i+1} (subscripts modulo n) for $0 \le i \le n-1$.

The *wheel* graph W_n has n vertices $v_1, \dots, v_n$, with single edges joining v_1 to v_i for $2 \le i \le n$, and v_i to v_{i+1} for $2 \le i \le n-1$, and v_n to v_2.

A *polyhedron graph* is the graph formed by the vertices and edges of a polyhedron. Example is a tetrahedron.

The *complete bipartite graph* $K_{m,n}$ is the bipartite graph of order $m + n$ with bi-parts $V = \{v_1, \dots, v_m\}$ and $W = \{w_1, \dots, w_n\}$, and single edges joining v_i to w_j for $1 \le i \le m$ and $1 \le j \le n$. A complete multi-partite graph (with any number of parts) can be defined similarly. The graph $K_{1,n}$ is sometimes called a *claw graph*, or a *star graph.*

1.3. **Paths and connectedness.** A *walk of length* m from a vertex v_0 to a vertex v_m in a graph G is a sequence of vertices $(v_0, \ldots, v_m)$ such that v_i is adjacent to v_{i+1} for $0 \leq i < m$. A walk having distinct vertices is called a *path.* A *closed walk* is a walk from a vertex to itself, that is, one with the same initial vertex as final vertex, and a *circuit* is a closed walk with distinct vertices, except (obviously) for the initial and final ones. A circuit of length k is called a k*-cycle.*

The *distance* between two vertices v and w in a graph G is the length of the shortest path from v to w, and is denoted by $d_G(v, w)$. The *diameter* of G is the maximum distance between a pair of vertices. The *girth* of a graph is the length of its shortest non-trivial circuit (if there is one). For example, the girth of K_n is 3 for all $n \geq 3$; the girth of C_n is n for all n; the girth of $K_{m,n}$ is 4 if $m, n \geq 2$.

A graph G is said to be *connected* if it has the property that given any two vertices v and w of G there is a path in G from v to w. Otherwise G is said to be *disconnected.*

The *connected component* containing a given vertex v is the subgraph of G consisting of all the vertices w for which there is a path in G from v to w, together with all the edges in G which are incident to any such vertex. Connected components partition the graph G into disjoint subgraphs whose union is G.

Theorem 1.2. *Let G be a simple graph of order n. If G has k connected components, then* $n - k \leq |E(G)| \leq \binom{n-k+1}{2}$.

Proof. We firstly show $n - k \leq |E(G)|$ by strong induction on $|E(G)|$. For the base case $|E(G)| = 0$, this implies that G consists of n components, so $k = n$. This clearly holds. Now, removing an edge from G, either k stays the same or increases by 1. This gives us the result rather easily.

Suppose the maximum number of edges is obtained when components of sizes $n_1 \leq n_2 \leq \ldots \leq n_k$. Suppose some n_i with $1 \leq i \leq k$ is greater than 1. Remove at most $n_i - 1$ edges incident to vertex v in component n_i, and add v to the component n_{i+1}, adding $n_{i+1} - (n_i - 1) > 0$ edges. So, we increased the number of edges, a contradiction. This means that the maximum is achieved when $n_1 = n_2 = \ldots = n_{k-1} = 1$, and $n_k = n - k + 1$. This gives us desired result. □

1.4. **Euler paths/tours and Hamiltonian cycles.** An *Euler path* in a graph G is a walk which includes every edge of G exactly once. If such a path exists, then the graph is said to be *semi-Eulerian.* An *Euler tour* is a closed Euler path, and any graph which has an Euler tour is called *Eulerian.*

Lemma 1.3. *Every finite graph whose vertices all have degree at least* 2 *contains a circuit.*

Theorem 1.4. (Euler's Theorem). *A connected finite graph has an Euler tour if and only if each of its vertices have even degree.*

Proof. Suppose there exists a non-trivial closed walk in G. Then the removal of the edges of this walk leaves a union of connected components, all of which can be assumed to have an Euler tour by induction on the edges.

We use induction on the number of edges. The result holds for connected graphs with $0, 1, 2$, or 3 edges (and all vertices of even degrees). These graphs are K_1, K_3, singleton graph with loop, and graph on two vertices with parallel edges.

We now show G has a non-trivial circuit, when $|E(G)| \geq 3$. Let v_0 be any vertex of G and v_1 be adjacent to v_0 (exists because G is connected and $|E(G)| \geq 3$). Let

v_2 be adjacent to v_1, and so on until eventually $v_i = v_j$ for some $i < j$. Hence, there exists a closed walk and thus we have a circuit. □

Corollary 1.5. *A connected finite graph has an Euler path if and only if it has at most two vertices of odd degree.*

A *Hamiltonian path* in a graph is a path which passes through every vertex exactly once. If such a path exists, the graph is said to be *semi-Hamiltonian*. A *Hamiltonian cycle* (or *Hamiltonian circuit*) is a circuit passing through each vertex exactly once, and any graph which has a Hamiltonian cycle is called *Hamiltonian*.

An example of a Hamiltonian graph is the cube Q_3. Non-Hamiltonian graphs include those with cut vertices and bridges (example of former are star graphs on at least three vertices).

Theorem 1.6. (Ore's Theorem). *If G is a simple graph with $n \geq 3$ vertices, and has the property that the sum of the degrees of every two non-adjacent vertices v and w is always at least n, then G is Hamiltonian.*

Proof. To derive a contradiction, assume this is false, and let G be a graph with the given properties such that G does not have a Hamiltonian cycle. Add as many vertices, each adjacent to all others, to create a larger graph H of order $n+k$ (such that k is as small as possible) that is Hamiltonian. Say $v_1 \dots v_{n+k}$ is a Hamiltonian cycle in H.

Suppose $v_1 \in V(G)$ and $v_2 \notin V(G)$. So, without loss of generality, $v_3 \notin V(G)$, for otherwise k could be smaller. We may further assume v_1 and v_3 are non-adjacent, since we could otherwise remove v_2. Therefore, $\deg(v_1) + \deg(v_3) \geq n$. Suppose v_1 is adjacent to v_i for some $v_i \in V(G)$. If $\{v_1, v_i\} \in E(G)$, then $\{v_3, v_{i+1}\} \notin E(G)$, since we could otherwise create a new cycle by following v_1 to v_i, followed by v_i to v_3, then v_3 to v_{i+1} and finally v_{i+1} to v_{n+k} (so we could remove v_2). Hence, for every neighbour v_i of v_1 in G, we have a non-neighbour of v_3, which must be in G since all vertices of $H \setminus G$ are adjacent to all vertices in G. Note this holds for $4 \leq i \leq n+k$. Let $r = \deg_G(v_1)$ and $s = \deg_G(v_3)$. Then $r + s \geq n$ by hypothesis. But also the number of vertices in G which are not adjacent to v_3 is $n - s - 1$, so $r \leq n - s - 1$. But then $r + s \leq n - 1$, a contradiction. □

Corollary 1.7. (Dirac's Theorem). *If G is a simple graph with $n \geq 3$ vertices such that every vertex has degree at least $\frac{n}{2}$, then G is Hamiltonian.*

Proof. The sum of the degrees of every two vertices is at least $\frac{n}{2} + \frac{n}{2} = n$. □

Note. Both theorems more or less say that if G has "enough" edges, then it is Hamiltonian. But non-Hamiltonian graphs can still have many edges. For example, taking K_{n-1} and a single vertex with a single edge connecting the two subgraphs (an isthmus) is non-Hamiltonian.

1.5. **Trees.** A *tree* is a connected graph with no circuits. A *forest* is a graph with no circuits (union of trees).

Every tree is *bipartite*.

Proposition 1.8. *A graph G is bipartite iff it has no closed walks of odd length.*

Proof. ($\Longrightarrow$) Suppose G is bipartite. Then any closed walk alternates between the two partition sets, and it must come back to original vertex, so it cannot be odd.

($\Longleftarrow$) Suppose G has no closed walks of odd length. Let v_0 be any vertex of G. We assume G is connected. Then the sets A (B) of vertices of G in which the shortest path between $v \in A$ (B) and v_0 is of even (odd) length partition G. Moreover, there is no edge between the sets A and B, since if there were, one could construct a closed walk of odd length. □

Theorem 1.9. *If T is a graph of order n, then the following are all equivalent:*

(1) *T is a tree;*
(2) *T has $n-1$ edges, but no circuits;*
(3) *T is connected and has $n-1$ edges;*
(4) *T is connected, and each of its edges are a bridge;*
(5) *any two vertices of T are connected by exactly one simple path;*
(6) *T has no circuits, but the addition of any new edge creates exactly one circuit.*

Proof. ((1) $\implies$ (2)). Suppose T is a tree. Then we know T has no circuits, and it is also connected. Proof by induction on $|E(T)|$. Clearly true for 0 or 1 edges.

Remove an edge. Every edge is an isthmus, since otherwise we could construct a circuit. So we get two subgraphs T_1 and T_2 which are trees. By the inductive hypothesis and summing their edges, we get $n-1$ is the number of edges in T.

((2) $\implies$ (3)). Need to show T is connected; we know it has $n-1$ edges but no circuits. Let k be the number of connected components of T, and suppose these components have order $n_1, \dots, n_k$ with $n_1 + \dots + n_k = n$. By previous argument, each T_i has $n_i - 1$ edges. So $|E(T)| = \sum_{1 \le i \le k} n_i - 1 = n - k$, so $k = 1$ implies T is connected.

((3) $\implies$ (4)). We know T is connected and has $n-1$ edges. Let e be any edge. If e is not an isthmus, then $T \setminus \{e\}$ is connected but has only $n-2$ edges. This contradicts a previous theorem (lower bound on number of edges).

((4) $\implies$ (5)). Existence of the path follows from connectedness. Uniqueness: if there are two such paths, then the removal of any edge e leaves the graph connected, a contradiction.

((5) $\implies$ (6)). If we suppose there is a circuit, one can get two paths by splitting the circuit. Suppose we add an edge. Then we get a circuit rather trivially, unique by similar argument.

((6) $\implies$ (1)). If T were not connected, then adding an edge between the two components would not create a circuit. □

Corollary 1.10. *Every finite tree with more than one vertex has leaves (a leaf is a vertex of degree* 1*).*

Proof. Apply the Handshaking Lemma. □

Corollary 1.11. *A forest with n vertices and k components has $n-k$ edges (hence a forest meets the lower bound on number of edges in earlier theorem).*

Proof. Apply the Handshaking Lemma. □

1.6. **Spanning tree.** A *spanning tree* is a subgraph of a graph G which is a tree and contains every vertex. Similarly, a *spanning forest* is a subgraph of a graph G which is a forest and contains every vertex.

Theorem 1.12. (Cayley's Theorem). *The number of spanning trees for K_n is n^{n-2}. Equivalently, the number of vertex labelled trees on n vertices is n^{n-2}.*

Proof. Let T_k be the number of vertex labelled trees on n vertices in which a given vertex v has degree k. Then the total number of vertex labelled trees on n vertices is $\sum_{k=1}^{n-1} T_k$. Note that $T_{n-1} = 1$.

We will now show that $(n-k)T_{k-1} = (k-1)(n-1)T_k$ for $2 \le k \le n$, or $1 \le k-1 \le n-1$.

Suppose S is a vertex labelled tree of order n in which the given vertex v has degree $k-1$ (where $1 \le k-1 \le n-2$). Then since S has $n-1$ edges but only $k-1$ of those are incident with v, there must be some other edge uw in S that is not incident with v. Removal of this edge uw disconnects S into two components, one which must contain v and its neighbours in S. Now, add back an edge from v to z. Then this is a spanning tree T of K_n, where v has degree k.

Conversely, suppose we had such a T in which v has degree k. Say we remove an edge incident to v, disconnecting the graph into two components. For any vertex adjacent to v, the removal of edge vz disconnects T into two components, one containing v and its neighbours and the other containing z. Now we add a new edge from one of the neighbours of v to z, giving a new spanning tree where v has degree $k-1$.

We now count the number of pairs (S,T) that can occur in each cases.

In the first case, we choose the edge wz in $(n-1)-(k-1)=n-k$ ways. So, total number if $(n-k)T_{k-1}$.

In the second case, let $z_1, z_2, \dots, z_k$ be the k neighbours of v in T. The removal of the edge vz_i leaves z_i in component C_i of size n_i such that $n = 1+n_1+\ldots+n_k$ and we choose w in $n-1-n_i$ ways. Hence, number of choices is

$$\sum_{1\le i\le k}(n-1-n_i) = k(n-1) - \sum_{1\le i\le k} n_i = k(n-1)-(n-1) = (k-1)(n-1),$$

and so total number of pairs is $(k-1)(n-1)T_k$.

We can now show by induction on j that $T_{n-j} = \binom{n-2}{j-1}(n-1)^{j-1}$ for $1 \le j \le n-1$.

For $j=1$, this holds because $T_{n-1}=1$. Now assume it holds for some $j \in \{1,2,\dots,n-2\}$. Then taking $k=n-j$, we find that

$$\begin{aligned} jT_{n-j-1} &= (n-j-1)(n-1)T_{n-j} \\ &= (n-j-1)(n-1)\binom{n-2}{j-1}(n-1)^{j-1} \\ &= \frac{(n-j-1)(n-2)!(n-1)^j}{(j-1)!(n-j-1)!} \\ &= \frac{(n-2)!(n-1)^j}{(j-1)!(n-j-2)!} \end{aligned}$$

and so

$$T_{n-j-1} = \frac{(n-2)!(n-1)^j}{j!(n-j-2)!} = \binom{n-2}{j}(n-1)^j,$$

which completes the induction.

The total number of spanning trees for K_n is therefore

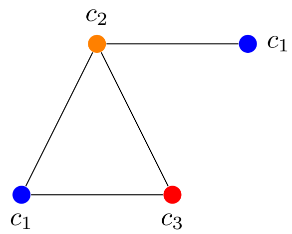
$$\begin{aligned} \sum_{1\le j\le n-1} T_{n-j} &= \sum_{1\le j\le n-1}\binom{n-2}{j-1}(n-1)^{j-1} \\ &= \sum_{0\le i\le n-2}\binom{n-2}{i}(n-1)^i \\ &= (1+(n-1))^{n-2} \\ &= n^{n-2}. \end{aligned}$$

□

1.7. **Graph colourings and maps.**

Definition 1.13. A graph G is called *k-colourable* (technically, *k-vertex-colourable*) if there exists a mapping $f: V(G) \to \{c_1, c_2, \dots, c_k\}$ (k colours) with the property that $f(v) \ne f(w)$ whenever v and w are two adjacent vertices.

Example 1.14. Below, the graph is 3-colourable.



Some other examples:

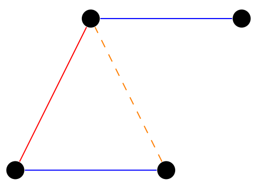
- N_n is 1-colourable;
- K_n is n-vertex colourable;
- C_n is
$$\begin{cases} \text{2-colourable,} & \text{if } n \text{ is even;} \\ \text{3-colourable,} & \text{if } n \text{ is odd.} \end{cases}$$
- Every bipartite graph is 2-colourable.

Definition 1.15. The *chromatic number* of a given graph G, denoted $\chi(G)$, is the smallest k such that G is k-vertex-colourable.

For example, $\chi(G) = 2$ whenever G is bipartite.

Definition 1.16. A graph is k-edge-colourable if its edges can be assigned up to k colours (by a colouring function $f : E(G) \to \{c_1, \ldots, c_k\}$) such that any two incident edges have different colours. The *edge-chromatic number* of a given graph G, denoted $\chi_e(G)$, is the smallest k such that G is k-edge colourable.

Example 1.17. For example, consider the graph below.

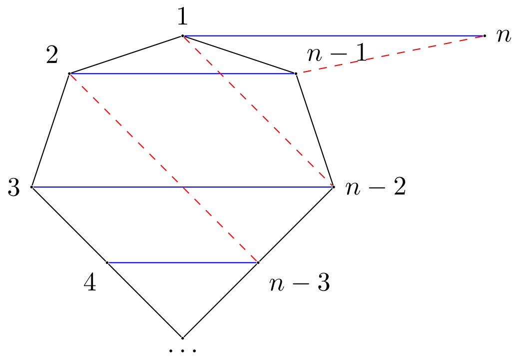


We also have that:

- $\chi_e(N_n) = 0$ for all n;
- $$\chi_e(C_n) = \begin{cases} 2 & \text{if } n \text{ is even;} \\ 3 & \text{otherwise.} \end{cases}$$
- $$\chi_e(K_n) = \begin{cases} n-1 & \text{if } n \text{ is even;} \\ n & \text{otherwise.} \end{cases}$$
- $\chi_e(K_{m,n}) = \max\{m, n\}$.

Proposition 1.18.
$$\chi_e(K_n) = \begin{cases} n-1 & \textit{if } n \textit{ is even;} \\ n & \textit{otherwise.} \end{cases}$$

Proof. Suppose n is even. Each vertex is incident with one edge of each colours using $n-1$ colours. So, we can colour as shown below (think of the ones corresponding to colouring the blue edges, then apply same principal through rotation of the diagram to get all others):



If n is odd, then we do the same trick for $n+1$ vertices, which is even, and remove its n incident edges. □

Note. K_n has $\frac{n(n-1)}{2}$ edges, and each vertex is incident with edges of at least $n-1$ colours. Therefore, the number of edges of a given colour is at most

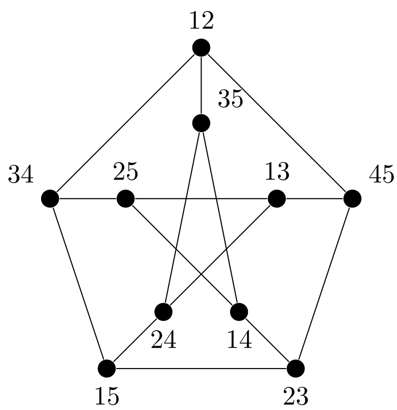
$$\frac{n-1}{2} \leq \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd;} \\ \frac{n-2}{2} & \text{if } n \text{ is even.} \end{cases}$$

Lemma 1.19. *If G is a simple finite graph such that $\deg_G(v) < k$ for all vertices v of G, then G is k-vertex-colourable.*

Proof. We provide a proof by induction on $|V(G)|$. Clearly the result holds for orders 1 and 2.

Let v be any vertex of G, and remove it including all incident edges, and we have a smaller graph H with $\deg_H(w) < k$ for all $w \in V(H)$. By induction, H is k-vertex-colourable. But $\deg_G(v) < k$, therefore there exists at least one of the k colours in H that is not used for vertices adjacent to v. So we can colour v with that colour and get a k-vertex colouring of G. □

The *Peterson graph*, shown below, can be represented as the the graph G with vertices represented as all subsets of $\{1, 2, 3, 4, 5\}$ of cardinality 2, with an edge between two vertices occurring iff they are set-disjoint:



and has the properties:

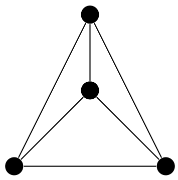
- 15 edges;
- 10 vertices;

- 3-regular;
- non-bipartite;
- has diameter 2;
- has girth 5 (has no three cycles or four cycles).

Suppose G is a connected simple 3-regular graph of diameter 2. The largest of $|V(G)|$ is 10.

1.8. **Maps.** A *map* is an embedding of a connected graph on some surfaces, such as the sphere (or equivalently the plane by stereographic projection), torus/klein bottle, projective plane and so on, without any *edge-crossings*.

We can embed K_4 in on the plane:



But we cannot embed K_5 on the sphere (note we can on the torus).

A graph is called *planar* if it can be drawn on the plane, and *non-planar* otherwise (and *torul* if it can be on torus but not on sphere).

We can embed C_n on the sphere using the *equatorial map* (send each vertex on the equator and join by edge), yielding n vertices, n edges and 2 faces.

An embedding of a multigraph on sphere using the *polar map* by taking vertices to be antipodal point and joining n edges between them (yielding 2 vertices, n edges and n faces).

Theorem 1.20. (Kuratowski's Theorem). *A finite connected non-planar graph has K_5 or $K_{3,3}$ as a subgraph.*

Proposition 1.21. (Euler's Polyhedral formula). *If M is a map in the plane/sphere with $|V|$ vertices, $|E|$ edges and $|F|$ faces, then*

$$|V| - |E| + |F| = 2.$$

Proof. We provide a proof by induction. Remove an edge e from the map. If e is an isthmus/bridge, then its removal gives two connected components, of which can be embedded in the plane/sphere in the map given by M. Then if these resulting maps have $|V_i|$ vertices, $|E_i|$ edges, and $|F_i|$ faces for $i = 1, 2$, then the EPF holds for them. Moreover, $|V_1| + |V_2| = |V|$, $|E_1| + |E_2| = |E| - 1$ and $|F_1| + |F_2| + 1$. It easily can be seen that the formula then holds.

If instead e is not a bridge, then we get $|V|$ vertices, $|E| - 1$ edges, and $|F| - 1$ faces, and again this clearly holds. □

Note. This can be generalised to the Euler's characteristic; $|V| - |E| + |F|$ will be constant for all maps in S and equal to Euler's characteristic function. We get 0 for torus, -2 for double torus, -4 for triple torus and so on, 1 for projective plane, 2 for Klein bottle and so on.

Lemma 1.22. *Every simple connected finite planar graph has a vertex of degree at most* 5.

Proof. Suppose this graph has n vertices, m edges, f faces, and embedded over the plane. Then $n - m + f = 2$ (by EPF). Now, count incident edge-face pairs in 2 different ways. Every face has at least 3 edges (since the graph is simple). Every edge has at most 2 faces. Hence, $3f \leq 2m$. But now if every vertex has degree ≥ 6, then by the Handshaking Lemma it follows $2m \geq 6n$, i.e., $m \geq 3n$. But then

$$2 = n - m + f \leq \frac{m}{3} - m + \frac{2m}{3} = m - m = 0.$$

□

Theorem 1.23. (Six-colour theorem). *Every simple connected finite planar graph has a* 6*-vertex-colouring.*

Proof. By induction on the number of vertices on graph. By Lemma, there exists a vertex v of G with degree at most 5. Remove that vertex and incident edges to yield a simple planar subgraph H, in which each component is 6-colourable by induction. And we can colour the removed vertex v with any colour other than the colour of its at most 5 neighbours in G. □

Theorem 1.24. *Every simple connected finite planar graph is* 5*-vertex colourable.*

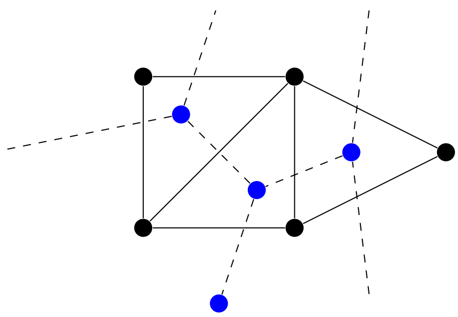
Proof. Again, G has a vertex v of degree at most 5. Remove v and its incident edges, leaving a 5-vertex-colourable subgraph H (by induction hypothesis).

If the neighbours of v in H use at most 4 colours, then there is a spare colour for v, making G 5-vertex-colourable. Hence, we suppose v has 5 neighbours in H, all coloured differently. Consider the subgraph H_{ij} consisting of all vertices coloured c_i and c_j. This is bipartite. Suppose there is no edge joining v_i and v_j. Then we can swap the colours of all vertices in the graph of H_{ij}. This removes a colour of H. But then we are using at most four colours for neighbours of v, so G has 5 vertex colouring.

If v_i and v_j belong to different components of H_{ij}, then there exists no edge from v_i to v_j and we can swap colours of vertices in H_{ij} in the component containing v_j, and obviously get a 5 colouring.

If this never happens, then for all $i \neq j$ the vertices v_i and v_j are in the same component of H_{ij}. Now there exists a path in H_{13} from v_1 to v_3 and a path in H_{24} from v_2 to v_4 and these paths must cross, contradicting G is planar. □

A *dual* of a map is one in which the faces become vertices, and edges are drawn between the vertices iff the two faces in the original graph meet at an edge (for each one). For example, consider the graph below:



Corollary 1.25. *Every connected planar map has a* 5*-colouring.*

Proof. Apply the above theorem to the dual. □

Theorem 1.26. (Four Colour Theorem). *Every simple finite planar graph is* 4*-vertex-colourable.*

Theorem 1.27. (Brooke's Theorem). *If G is a simple graph with maximum degree k, then G is k-vertex-colourable unless either one of its components is isomorphic to K_{k+1}, or $k = 2$ and one of its components is isomorphic to an odd cycle.*

Proof. Induction on $|V(G)|$. True for $k = 1$ (G is 1-vertex colourable iff G is null), and true for $k = 2$ (G is 2-vertex-colourable iff G is bipartite which happens iff it has no odd cycles). Without loss of generality, $k \geq 3$ and G is connected, otherwise it holds for each component by induction and hence holds for G. Therefore, we have to show G is k-vertex-colourable or is equivalent to K_{k+1}.

So, let v be a vertex of G and consider $H = G - v$. Then there exists vertices of H with degree less than k, implying H is not isomorphic to K_{k+1}, so by induction H is k-vertex-colourable. Next, if $\deg_G(v) < k$, then there exists a spare colour for v and so G is k-vertex-colourable. So, we may suppose $\deg_G(v) = k$. Without loss of generality, G is k-regular (since v is arbitrary).

Now, let $w_1, w_2, \ldots, w_k$ be the neighbours of v in G. If the given k-vertex-colouring of H uses less than k colours for $w_1, w_2, \ldots, w_k$, then again there exists a spare colour for v and so we may assume $w_1, w_2, \ldots, w_k$ are coloured differently (say $c_1, c_2, \ldots, c_k$, respectively). Next, let H_{ij} be the subgraph of H induced on the vertices coloured c_i and c_j. If w_i and w_j are in two different components of H_{ij}, then as in proof of the five colour theorem, we may swap colours in one of the components, and again get a k-vertex-colouring for G.

Hence, we may now suppose that for all $i \neq j$ the vertices w_i and w_j are in the same component of H_{ij}. Therefore, we may assume each w_i is adjacent to just one vertex colour c_j. So, for vertex z adjacent to w_i can be adjacent to only two vertices of colour c_i, namely w_i and the next on the path (for otherwise has at most $k - 3$ colours adjacent, so can be recoloured). Thus, the component H_{ij} is equal to the path P_{ij} (which holds for all $i \neq j$).

If i, j, ℓ are distinct, then the paths P_{ij} and $P_{i\ell}$ can only meet at w_i. For otherwise, we can recolour the vertex which is in both the paths. But now we can swap the colours of $P_{i\ell}$ (c_i and c_ℓ swapped).

Vertex z lies on the path P_{ij} and $P_{j\ell}$ now, because it is now coloured c_j and adjacent to vertex w_i now coloured c_ℓ.But then two paths intersect, and so previous case applies. Unless there exists no such z, i.e., each path P_{rs} is an edge, implying $H \cong K_k$ and thus $G \cong K_{k+1}$. □

Theorem 1.28. Vizing's Theorem. *If G is a simple finite graph with maximum degree k, then $k \leq \chi_e(G) \leq k + 1$.*

If $\chi_e(G) = k$, then we say it is *class* 1. Otherwise, if $\chi_e(G) = k + 1$, then we say it is *class* 2. So K_n is class one when n is even, and class two when n is odd.

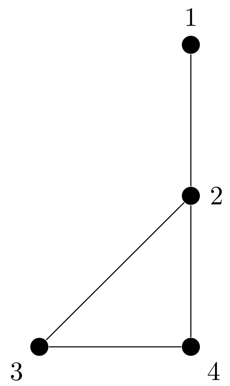
If maximum degree is 2, then G is a union of paths and cycles, and then G is class 1 iff it is bipartite, or equivalently, G is class 2 iff it contains a cycle of odd length. So G with max degree 2 has $\chi_e(G) = 2$ if there exists no cycle, or all cycles have even length, and $= 3$ otherwise.

2. Spectral Graph Theory

Suppose G is simple/finite. The *adjacency matrix* $A = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E(G); \\ 0 & \text{otherwise.} \end{cases}$$

For example, the graph below

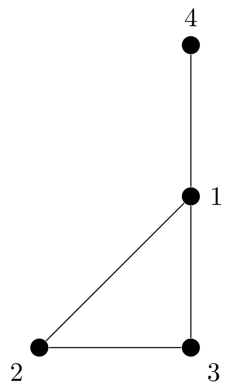


has adjacency matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Firstly, observe adjacency matrices for simple/finite undirected graphs are symmetric. Also, the trace of adjacency matrices is 0 (due to simple condition). Note that the adjacency matrix depends on the ordering of the vertices.

Any re-ordering of the vertices of $V(G)$ has the adjacency matrix become a conjugate of A as the new adjacency matrix. Indeed, $P^T AP = P^{-1}AP$ where P is a permutation matrix. For example,



has adjacency matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

determined by the permutation $(1, 4, 3, 2)$ (read from right to left).

Note: if x is an eigenvector for the matrix A, with eigenvalue λ, then $xA = \lambda x$. So, $(xP)\big(P^{-1}AP\big) = xAP = \lambda xP$. Hence, λ is an eigenvalue for $P^T AP$ with eigenvector xP. Also, multiplicity of λ same for both.

Theorem 2.1. *Let A be an adjacency matrix for G with vertex set $\{v_1, v_2, \ldots, v_n\}$. Then for each non-negative integer k, the number of walks in G of length k from v_i to v_j is $[A^k]_{ij}$.*

Proof. By induction, easy for $k = 0$ (number of walks of length 0 from v_i to v_j is 0 if $i \neq j$ and 1 if $i = j$). For $k = 1$, simply consider the edges.

Induction step:

$$\begin{aligned}
[A^{k+1}]_{ij} &= \sum_{\ell=1}^{n} [A^k]_{i\ell}[A]_{\ell j} \\
&= \sum_{\ell=1}^{n} (\text{number of walks of length } k \text{ from } v_i \text{ to } v_\ell) \times (\text{number of edges of } v_\ell \text{ to } v_j) \\
&= \sum_{\ell=1}^{n} (\text{number walks of length } k+1 \text{ from } v_i \text{ to } v_j).
\end{aligned}$$

□

Corollary 2.2. $\operatorname{Trace}(A^2) = 2|E|$.

Proof. Observe

$$\begin{aligned}
\operatorname{Trace}(A^2) &= \sum_{i=1}^{n} [A^2]_{ii} \\
&= \sum_{i=1}^{n} (\text{number walks length 2 from } v_i \text{ to } v_i) \\
&= \sum_{i=1}^{n} \deg(v_i) = 2|E|
\end{aligned}$$

by Hanshaking lemma. □

Corollary 2.3. *If k is the number of triangles in G, then $k = \frac{1}{6}\operatorname{Trace}(A^3)$.*

Proof. Observe

$$\begin{aligned}
\operatorname{Trace}(A^3) &= \sum_{i=1}^{n} (\text{number walks of length 3 from } v_i \text{ to } v_i) \\
&= \sum_{i=1}^{n} (\text{number oriented 3-cycles from } v_i \text{ to } v_i) \\
&= \text{number of 3-cycles } (v_i, v_j, v_\ell, v_i) \text{ in } G \\
&= (3!)(\text{number triangles in } G).
\end{aligned}$$

□

The eigenvalues of a finite simple graph are the eigenvalues of its adjacency matrix.

Properties:

- All eigenvalues are real, because A is real and symmetric.
- There exists an orthonormal basis $\mathcal{B}$ for $\mathbb{R}^n$ made up of eigenvectors for A.
- In fact, if Q is the $n \times n$ matrix with columns the vectors in $\mathcal{B}$, then Q is an orthogonal matrix (i.e., $Q^T Q = I_n$), since $x_i^T x_j = 1$ if $i = j$ and 0 otherwise (where $x_i, x_j \in \mathcal{B}$). Therefore, $Q^{-1} = Q^T$, and also $Q^T A Q = D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ where λ_i are corresponding eigenvalues. For the jth

column of $AQ = Ax_j = \lambda_j x_j$, therefore the (i,j)th entry of $Q^T AQ$ is

$$\begin{aligned} x_i^T A x_j &= x_i^T \lambda_j x_j \\ &= \lambda_j \big(x_i^T x_j\big) \\ &= \begin{cases} \lambda_j & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned}$$

Note that

$$\operatorname{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

The multiset consisting of the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ with their multiplicities is called the *spectrum* of G (and of A).

Note that for square matrices B and C, $\operatorname{Trace}(BC) = \operatorname{Trace}(CB)$.

- We have

$$\operatorname{Trace}(A) = \operatorname{Trace}\big(QDQ^T\big) = \operatorname{Trace}\big(DQ^TQ\big) = \operatorname{Trace}(D) = \lambda_1 + \ldots + \lambda_n.$$

- Similarly,

$$\operatorname{Trace}\big(A^k\big) = \operatorname{Trace}\big(QD^kQ^T\big) = \operatorname{Trace}\big(D^k\big) = \lambda_1^k + \ldots + \lambda_n^k.$$

Hence, $\lambda_1 + \ldots + \lambda_n = 0$, $\lambda_1^2 + \ldots + \lambda_n^2 = 2|E|$ and so on.

Definitions for eigenvalues and eigenspaces for G see slides/notes.

Eigenvectors can be viewed as labellings of the vertices of G with a special property:

Suppose $x = (x_1, \dots, x_n)$ is an eigenvector for G (i.e., for A) with eigenvalue λ. Label the ith vertex with entry x_i of the vector x. Then

$$\sum_{j \text{ s.t. } v_j \text{ is adjacent to } v_i} x_j = \text{single row entry } i\text{th row of } Ax = [Ax]_i = [\lambda x]_i = \lambda x_i.$$

That is, λ times the label of any vertex is the sum of the labels of its neighbours. Conversely, any labelling with this property gives an eigenvector for G with eigenvalue λ, since $[Ax]_i = \lambda x_i$ implies $Ax = \lambda x$.

If $\sum_{v_j \sim v_i} x_j = \lambda x_i$, λ constant, then $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is an eigenvector for G with eigenvalue λ.

Corollary 2.4. *If G is a d-regular graph then d is an eigenvalue for G with constant eigenvector* $v = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

Proof.

$$\sum_{v_j \sim v_i} x_j = \sum_{v_j \sim v_i} 1 = d.$$

□

Lemma 2.5. *If $G = G_1 \cup G_2$ (disjoint union) then* $\operatorname{Spec}(G) = \operatorname{Spec}(G_1) \cup \operatorname{Spec}(G_2)$.

Proof. Is easy,

$$A(G) = \begin{pmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{pmatrix}$$

and

$$Q^T A Q = D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

where D_i is diagonal with eigenvalues of G_i in diagonal. □

Theorem 2.6. *A graph G is bipartite iff its spectrum is symmetric about* 0.

Proof. ($\Longrightarrow$) Let $G = U \cup W$ be the bipartite separation.

$$\sum_{v_j \sim v_i} x_j = \lambda x_i,$$

so

$$\sum_{v_j \sim v_i} (-x_j) = -\sum_{v_j \sim v_i} x_j = -\lambda x_i$$

for $x_j \in W$. Now,

$$\sum_{v_j \sim v_i} x_j = \lambda x_i = (-\lambda)(-x_i)$$

for $x_i \in U$. Hence, $-\lambda$ is an eigenvalue of G too.

($\Longleftarrow$) Suppose the spectrum of G is symmetric about 0 (i.e., $\lambda \in \mathrm{Spec}(G)$ iff $-\lambda \in \mathrm{Spec}(G)$). Then for each $k \in \mathbb{N}$, $\mathrm{Trace}\big(A^k\big)$ is the sum $\sum_{i=1}^{k} \lambda_i^k$ and equals $\sum_{i=1}^{n} (-\lambda_i)^k$. That is,

$$\mathrm{Trace}\big(A^k\big) = (-1)^k \mathrm{Trace}\big(A^k\big).$$

If k is odd, this implies that the trace of A^k is 0. Thus, the number of walks of odd length is 0, and so G is bipartite. □

Theorem 2.7. (Bounds on eigenvalues). *Let λ be an eigenvalue of a simple finite graph G. Then*

(1) $|\lambda| \leq \max_{v \in V(G)} \deg(v) = \Delta(G)$.
(2) *This upper bound from* (1) *is attained iff some component of G is d-regular where $d = \Delta(G)$, and in that case d is an eigenvalue for G.*
(3) *In particular, if G is connected and $|\lambda| = \Delta(G)$ for some eigenvalue λ, then G is d-regular (with $d = \Delta(G)$) and d is an eigenvalue for G with multiplicity* 1.

Proof. Let $x = (x_1, x_2, \ldots, x_n)$ be an eigenvector for λ. Then

$$|\lambda||x_j| = |\lambda x_j| = |[Ax]_j| = \left| \sum_{v_i \sim v_j} x_i \right| \leq \sum_{v_i \sim v_j} |x_i|$$

for each j. Now, choose j such that $|x_j|$ is as large as possible (so $|x_i| \leq |x_j|$ for all i). Then

$$|\lambda||x_j| \leq \sum_{v_i \sim v_j} |x_i| \leq \sum_{v_i \sim v_j} |x_j| \leq d|x_j|.$$

Since x is an eigenvector, it is non-zero and so $|x_j| > 0$, implying $|\lambda| \leq d$ (proving (1)).

Suppose the bound is attained. Then

$$\left| \sum_{v_i \sim v_j} x_i \right| = \sum_{v_i \sim v_j} |x_i| = d|x_j|$$

as it follows v_j has d neighbours. Moreover, $|x_i| = |v_j|$ for all $v_i \sim v_j$, and in fact $x_i = x_j$ for all $v_i \sim v_j$ for otherwise $\left|\sum_{v_i \sim v_j} x_i\right|$ would be smaller than $d|x_j|$. Hence, the labels of all the neighbours of v_j are equal to the label on v_j. We can now repeat the argument and find that x is constant on the connected component of G containing v_j, and so is d-regular (and repeating argument constant on each component).

If G is connected, then G itself is d-regular, with $(x_j, x_j, \ldots, x_j)$ as eigenvector corresponding to eigenvalue d. Moreover, this argument shows every eigenvector for the eigenvalue d is a constant vector equal to $x_j(1, 1, \ldots, 1)$. There is only one up to scalar multiplication, therefore d has multiplicity 1. □

Corollary 2.8. *If G has m connected components and the maximum is attained on each component, then $d = \max_{v \in V(G)} \deg(v)$ is an eigenvalue with multiplicity equal to m.*

Theorem 2.9. *If G is a simple finite graph then the largest eigenvalue of G is $\geq$ average degree of the vertices of G.*

Proof. Let J be the column vectors of 1s in $\mathbb{R}^n$. Then $J^T A J = \sum_{i=1}^n \deg(v_i)$. Next, let $\mathcal{B}$ be an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors for G. Then since $J \in \mathbb{R}^n$, we know that

$$J = \sum_{x \in \mathcal{B}} c_x x$$

as a linear combination of vectors in $\mathcal{B}$. Let λ_x be the eigenvalue of G corresponding to the eigenvector x, for each $x \in \mathcal{B}$. Then for each $x \in \mathcal{B}$, we have

$$J^T x = \left(\sum_{y \in \mathcal{B}} c_y y\right)^T x = \sum_{y \in \mathcal{B}} c_y \left(y^T x\right)$$

and since $\mathcal{B}$ is orthonormal,

$$\delta_{xy} = y^T x = \begin{cases} 1 & \text{if } y = x; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$J^T x = \sum_{y \in \mathcal{B}} c_y \delta_{xy} = c_x.$$

Also,

$$J^T J = J^T \sum_{x \in \mathcal{B}} c_x x = \sum_{x \in \mathcal{B}} c_x \left(J^T x\right) = \sum_{x \in \mathcal{B}} = \sum_{x \in \mathcal{B}} c_x^2 = n.$$

Hence,

$$J^T A J = J^T A \left(\sum_{x \in \mathcal{B}} c_x x\right) = \sum_{x \in \mathcal{B}} c_x J^T A x = \sum_{x \in \mathcal{B}} J^T c_x \lambda_x x = \sum_{x \in \mathcal{B}} c_x \lambda_x c_x \leq \lambda n,$$

if λ is maximum eigenvalue. Hence,

$$\lambda \geq \frac{1}{n}\left(J^T A J\right) = \frac{1}{n} \sum_{v \in V(G)} \deg(v),$$

as desired. □

Lemma 2.10. *Suppose M is a real symmetric matrix such that $\left\{I, M, M^2, \ldots, M^D\right\}$ is linearly independent over $\mathbb{R}$. Then M has at least $D+1$ distinct eigenvalues.*

Proof. We know $Q^T MQ$ is a diagonal matrix with eigenvalues as entries, i.e,

$$\text{diag}(\lambda_1, \dots, \lambda_n) = Q^T MQ = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

for some orthogonal matrix Q, with $\lambda_1, \dots, \lambda_n$ as eigenvalues of M. We can arrange things so that the first k of these are distinct while the other $n-k$ are repetitions. Hence, the minimal polynomial of M is $m_M(x) = (x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_k)$ and so by the Caley-Hamilton theorem (which says that a matrix satisfies its min polynomial) we know that $0_{n\times n} = (M-\lambda_1 I_n)\dots(M-\lambda_k I_n)$. The latter can be written as a linear combination of $I, M, M^2, \dots, M^k$, and by linear independence of $I, M, \dots, M^D$ we know that $k > D$. Therefore, $k \geq D+1$, therefore M has at least $D+1$ distinct eigenvalues. □

Corollary 2.11. *Let G be a simple/finite connected graph. Then the number of distinct eigenvalues of G is at least* $\text{Diam}(G)+1$ *(where we recall* $\text{Diam}(G) = \max_{u,v\in V(G)} d(u,v)$*, i.e., longest shortest path in G).*

Proof. Let A be the adjacency matrix for G, and let v_i and v_j be vertices of distance $D = \text{Diam}(G)$ from each other. Then $[A^k]_{ij} = 0$ for $0 \leq k < D$, while $[A^D]_{ij} > 0$. We'll show that $I, A, A^2, \dots, A^D$ are LI. So suppose $\sum_{i=0}^{D} a_i A^i = 0$ for some $a_i \in \mathbb{R}$. Then the (i,j)th entry of the LHS is $\sum_{k=0}^{D} a_k [A^k]_{ij} = a_D [A^D]_{ij}$ where $[A^D]_{ij} > 0$ and must be $[0_{n\times n}]_{ij}$ implying $a_D = 0$. Applying same argument to vertices at distance $D-1$ from each other and so on, yields $a_0 = \dots = a_D = 0$. □

Important observations:

If A is the adjacency matrix for the simple finite graph G, then the adjacency matrix for $\overline{G}$ (complement of G) is

$$J_n - I_n - A$$

where $n = |V(G)|$ and J_n is the $n \times n$ all 1's matrix.

Proof. Let $\overline{A}$ be the adjacency matrix for $\overline{G}$. Then if $i = j$,

$$\left[A + \overline{A} + I_n\right]_{ij} = [A]_{ij} + \left[\overline{A}\right]_{ij} + [I_n]_{ij} = 0 + 0 + 1$$

while if $\{v_i, v_j\} \in E(G)$, then $\dots + \dots + \dots = 1 + 0 + 0 = 1$ and otherwise $\dots + \dots + \dots = 0 + 1 + 0 = 1$. Therefore,

$$J_n = I_n + A + \overline{A}$$

and simple rearrangement yields desired result. □

Example 2.12. $G = K_n$. Then $A = J_n - I_n$ and $\overline{A} = 0_{n\times n}$. What are the eigenvalues and eigenspaces?

Well,

$$J_n^2 = J_n J_n = nJ_n.$$

Therefore,

$$J_n^2 - nJ_n = 0_{n\times n}.$$

Now, let λ be any eigenvalue of J_n with eigenvector x. Then $\lambda x = J_n x = (s, s, \dots, s)$ where s is the sum of entries of x. For example, take $x = (1, 1, \dots, 1)$. So, $J_n x = nx$. All other eigenvalues lie in the orthogonal component of $\langle x \rangle$ by orthonormal basis property. Therefore, if x is an eigenvector for another eigenspace then $0 = J^T x = [s]$ where J is vector all entries 1 and s is the sum of entries in x. Therefore, $s = 0$, implying λx is the zero vector. But x is not the zero vector, so λ

must be 0. Hence, only eigenvalues of J_n are n and 0, where n has multiplicity 1 and 0 has multiplicity $n-1$. In fact, the vectors

$$\begin{pmatrix}1\\-1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}1\\0\\\vdots\\0\\-1\end{pmatrix}$$

are eigenvectors for eigenvalue 0 and these $n-1$ vectors are LI.

So, $\operatorname{Spec}(J_n) = \left\{n, \overbrace{0, \dots, 0}^{n-1}\right\}$. Also, we have $\operatorname{Spec}(I_n) = \{1, 1, \dots, 1\}$ with multiplicity n. Therefore, $\operatorname{Spec}(K_n) = \operatorname{Spec}(J_n - I_n) = \{n-1, -1, -1, \dots, -1\}$ with -1 having multiplicity $n-1$ with the same eigenspaces as J_n, i.e.,

$$\left\langle \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} \right\rangle, \text{ and } \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i = 0 \right\}.$$

Theorem 2.13. *Suppose we take a simple k-regular finite graph G of order n with* $\operatorname{Spec}(k, \lambda_2, \dots, \lambda_n)$. *Then* $\operatorname{Spec}(\overline{G}) = \{n-k-1, -1-\lambda_n, -1-\lambda_{n-1}, \dots, -1-\lambda_2\}$. *Moreover, if G and $\overline{G}$ are both connected, then G and $\overline{G}$ have the same eigenvectors.*

Proof. We know the vector J with all entries 1 is an eigenvector for G with eigenvalue k (since G is k-regular). Similarly, $\overline{G}$ is $(n-k-1)$-regular, with $n-k-1$ as eigenvalue and J as eigenvector.

Suppose x is any other eigenvector (not in $\langle J \rangle$) with eigenvalue $\lambda = \lambda_i$, and suppose x lies in an orthonormal basis $\mathcal{B}$ for $\mathbb{R}^n$ containing J. Hence, $J^T x = 0$, and so

$$\overline{A}x = (J_n - I_n - A)x = 0 - x - \lambda_i x = (-1 - \lambda_i)x.$$

Therefore, x is an eigenvector for $\overline{A}$ with eigenvalue $-1-\lambda_i$. The rest follows easily. Need connectedness on $\overline{G}$ to ensure that the eigenspace for $n-k-1$ has dimension 1. ☐

2.1. **Eigentheory of the Peterson Graph.** Firstly, note that the Peterson graph P is symmetric (and 3-arc-transitive). Now, let A be the adjacency matrix (for any ordering of the vertices). Then $[A^2]_{ij}$ is the number of walks of length 2 from v_i to v_j for which

$$\left[A^2\right]_{ij} = \begin{cases} 3 & \text{if } i = j; \\ 0 & \text{if } v_i \sim v_j \text{ is an edge;} \\ 1 & \text{if } i \neq j \text{ and } v_i \text{ is not a neighbour of } v_j. \end{cases}$$

Therefore,

$$A^2 = 3I_{10} + 0A + 1(J_{10} - I_{10} - A) = J_{10} - A + 2I_{10}.$$

Hence, $J_{10} = A^2 + A - 2I_{10}$. Note that all the 1's vector J is an eigenvector for J_{10} and A, with eigenvalues 10 and 3, respectively, both of multiplicity 1. Every other eigenspace is orthogonal to $\langle J \rangle$, therefore every non-constant eigenvector has entry-sum 0. Now, let λ be any other eigenvalue of A with eigenvector x. Then

$$\begin{aligned} J_{10}x &= \left(A^2 + A - 2I_{10}\right)x \\ &= A(Ax) + Ax - 2x \\ &= \lambda^2 x + \lambda x - 2x \\ &= \left(\lambda^2 + \lambda - 2\right)x. \end{aligned}$$

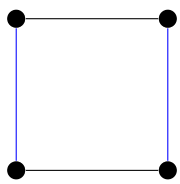
Therefore, x is an eigenvector for J_{10} with eigenvalue $\lambda^2 + \lambda - 2$. Therefore, $\lambda^2 + \lambda - 2 = 0$ (note if $\lambda^2 + \lambda - 2 = 10$, then $\lambda = 3$; recall J_{10} has eigenvalues $n = 10$ and 0). Hence, $(\lambda + 2)(\lambda - 1) = 0$, so $\lambda = -2$ or $\lambda = 1$. Next, let f and g be the multiplicities of 1 and -2, respectively. Note if on e of them is not an eigenvalue then multiplicity will be 0. Now, $1 + f + g$ is the sum of the eigenvalue multiplicities and consequently equals $n = 10$. Also, $(1 \times 3) + (f \times 1) + (g \times (-2))$ is the sum of the eigenvalues which is the trace of A which is 0. Therefore, $f + g = 9$ and $f - 2g = -3$ implying $3g = 12$ so $g = 4$ and $f = 5$. Thus, eigenvalues of the Peterson graph are 3, -2, and 1 with multiplicities 1, 4 and 5.

3. Graph Factorisation

A 1-*factor* in a simple graph G is a 'perfect matching' for G - that is, a set M of edges of G such that every vertex of G is incident with exactly one edge in M.

A 1-*factorisation* of a simple graph G is a collection of 1-factors with the property that every edge lies in exactly one of them.

Example 3.1. Consider C_4 below:



We also have Q_3 with three 1-factors and K_4 with three 1-factors.

Properties of 1-factors and 1-factorisation:

- If G has a 1-factor M, then $|V(G)|$ must be even because M has $\frac{|V(G)|}{2}$ edges.
- If G has a 1-factorisation, with say k 1-factors, then every vertex lies in all k of the 1-factors, but just one each, therefore has valency k. That is, G is k-regular.
- A 1-factorisation is a proper k edge colouring such that every vertex is incident with exactly one edge of each colour.
- A 1-factor is isomorphic to $\frac{n}{2}K_2$ (the union of $\frac{n}{2}$ copies of K_2.)
- A 1-factorisation is a "factorisation" of G into k copies of $\frac{n}{2}K_2$ (where $\frac{nk}{2} = |E(G)|$).

A *factorisation* of a given simple graph G into copies of a given simple graph H is a colouring $f : E(G) \to C$ of the edges of G with the property that the subgraphs of G induced by the sets of each colour $c \in C$ are all isomorphic to H.

Example 3.2. (1) The number of 1-factorisation of $K_{n,n}$ is the number of Latin squares of order n with a fixed first column. Simply label vertex sets

$\{1, \ldots, n\}$ and $\{w_1, \ldots, w_n\}$, then construct matrix below

$$\begin{pmatrix} w_1 & w_j & w_i & \ldots \\ w_2 & \ldots & & \\ \vdots & & & \\ w_n & \ldots & & \end{pmatrix}$$

such that each w_t occurs in exactly one row/column (i.e., is Latin square).

(2) As an exercise, find the 1-factors of K_6.

(3) The Peterson graph has no factorisation into three edge-disjoint 5-cycles. Remove any 5-cycle, then what remains can't be the union of two cycles, since there will be vertices of degree 1.

(4) Peterson graph has no 1-factorisation, since otherwise could give a 3-edge colouring (each 1-factor would have 5 edges).

(5) K_{10} is not factorisable into three copies of the Peterson graph.

Proof. Assume A does have one, and let P, Q and R be the adjacency matrices for the copies of the Peterson graph. Then

$$J_{10} = I_{10} + P + Q + R.$$

Next, J (the all 1's vector) is an eigenvector for all of J_{10}, I_{10}, P, Q and R with eigenvalues 10, 1, 3, 3 and 3. Now consider the eigenspaces for the eigenvalues -2 and 1 for each of P, Q and R. Observe that the eigenspaces for eigenvalue 1 of P and Q, say P_1 and P_1, both have dimension 5 and since $5 + 5 = 10 > 9$, it follows that P_1 and Q_1 overlap. Let x be an eigenvector for both P and Q with eigenvalue 1. Then

$$Rx = (J_{10} - I_{10} - P - Q)x = 0x - x - x - x = -3x,$$

and therefore x is an eigenvector for R with eigenvalue -3. But the Peterson graph has no eigenvalue -3, and thus we obtain a contradiction. □

4. Strongly regular graphs

Example 4.1. Peterson graph:

- 10 vertices;
- valency 3;
- Every 2 adjacent vertices have no common neighbours (no triangles);
- Every two non-adjacent vertices have 1 common neighbour (no four cycles).

So has SRG$(10, 3, 0, 1)$.

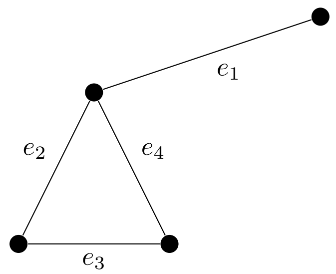
If G is a simple k-regular graph of order n, and there exist $\lambda, \mu \in \mathbb{Z}$ such that every two adjacent vertices of G have exactly λ common neighbours, and every two non-adjacent vertices of G have exactly μ common neighbours, then G is called a *Strongly regular graph* with parameters (n, k, λ, μ).

Examples:

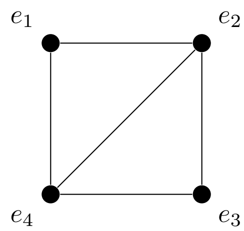
- C_5 has parameters $(5, 2, 0, 1)$ since 2-regular with order 5, and no triangles so $\lambda = 0$ and no four cycles so $\mu = 1$.
- K_n has parameters $(n, n-1, n-2, 0)$ since $n-1$-regular with order n, and each two adjacent vertices have all other $n-2$ vertices adjacent (also there are no non-adjacent vertices so $\mu = 0$).
- $K_{n,n}$ has parameters $(2n, n, 0, n)$ since $2n$ vertices, n regular, any two adjacent vertices have no common neighbours (bi-partite), and every two non-adjacent vertices belong to same partition so clearly n.

A *line graph* of a graph G has vertex set consisting of the edges of G with two edges adjacent iff they have a common vertex.

Example 4.2. Consider G below.



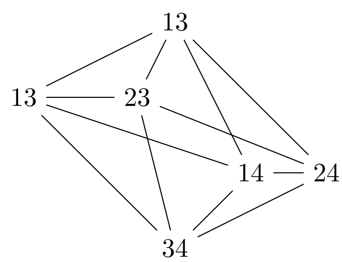
Then $L(G)$ is shown below.



Question: If G is a strongly regular graph, is $L(G)$ a strongly regular graph, and if so, what are its parameters?

Example 4.3. Triangular graphs for $n \geq 3$. Vertices are the 2-element subsets of $\{1, 2, \ldots, n\}$ with two such subsets joined by an edge iff they have exactly one element in common. Hence, $\overline{T(5)}$ is the Peterson graph.

We also have $\overline{T(4)}$ is the Octahedral graph:



$T(n)$ has parameters $\left(\frac{n(n-1)}{2}, 2(n-2), n-2, 4\right)$.

Square-lattice graphs $L_2(n)$ for $n \geq 2$. Vertices are the ordered pairs (a, b) of elements of $\{1, \ldots, n\}$ with two such pairs joined by an edge iff they agree in exactly one coordinate.

Paley graphs $P(q)$ for q a prime power equivalent to 1 mod 4. Vertices, are the exactly the elements joined by an edge iff their difference is a non-zero square in F_q. $P(5)$ has vertices $0, 1, \ldots, 4$ and non-zero square in $\mathbb{Z}_5$ are $1, 4 \equiv \pm 1$ so we get C_5 with 04321 cycle.

A lot of work is done on determining possibilities for parameters. We will look at two necessary conditions.

Theorem 4.4. *If there exists a SRG with parameters* (n, k, λ, μ) *then*
$$k(k - \lambda - 1) = (n - k - 1)\mu.$$

Proof. Let u be a vertex and v be a neighbour of u and w a neighbour of v such that $w \neq u$ and w not a neighbour of u. Question: How many possibilities are there for (v, w) given vertex u? On the one hand, k chpices for v and then $k - \lambda - 1$ for w so number pairs is $k(k - \lambda - 1)$. On the other hand, there are $n - k - 1$ choices for w and then μ choices for v, so $(n - k - 1)\mu$ and hence result. □

Theorem 4.5. (The Rationality Condition). *If there exists a connected SRG with parameters (n, k, λ, μ), then*

$$f, g = \frac{1}{2}\left(n - 1 \pm \frac{2k + (n-1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 - 4(k - \mu)}}\right).$$

Proof. First, suppose $\mu = 0$. Then any two non adjacent vertices lie in different components, and then also two vertices of the same component are adjacent. Therefore, G must be a complete graph isomoprhic to K_{k+1} and so $\lambda = k - 1 = n - 2$ and the rationality condition (TRC) holds.

So now suppose $\mu > 0$. Let A be an adjacency matrix for G. Then $[A^2]_{ij}$ is the number of walks length 2 from v_i to v_j which equals

$$\begin{cases} k & \text{if } i = j; \\ \lambda & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ \mu & \text{if } v_i \neq v_j \text{ are non-adjacent.} \end{cases}$$

Therefore,

$$A^2 = kI + \lambda A + \mu(J_n - I_n - A)$$

and so

$$\mu J_n = A^2 + (\mu - \lambda)A + (\mu - k)I_n.$$

If x is an eigenvector for A with eigenvalue ξ, then x is an eigenvector for J_n with eigenvalue $\frac{1}{\mu}\big(\xi^2 + (\mu - \lambda)\xi + (\mu - k)\big)$. Hence, either (a) x is a constant vector with eigenvalue k (since G is connected and k-regular), or (b) x is a vector with entry-sum 0 and so $J_n x = 0$.

In case (b),

$$\frac{1}{\mu}\big(\xi^2 + (\mu - \lambda)\xi + (\mu - k)\big) = 0$$

and so

$$\xi = \frac{-(\mu - \lambda) \pm \sqrt{(\mu - \lambda)^2 - 4(\mu - k)}}{2}.$$

Now let f and g be the multiplicities of these two possibilities for ξ. Then $1 + f + g$ is the sum of the multiplicities which equals n and

$$1 \times k + f\left(\frac{\lambda - \mu + \sqrt{\Delta}}{2}\right) + g\left(\frac{\lambda - \mu - \sqrt{\Delta}}{2}\right) = \text{Trace}(A) = 0.$$

We can aolve for f and g to get expression stated, and these must be non-negative integers because they are eigenvalue multiplicities. □

Example 4.6. Triangular graphs are also "Johnson graphs" which are m-subset of $\{1, \ldots, n\}$ if they intersect $m - 1$ elements.

Peterson graph is complement of $T(5)$.

C_5 has $(5, 2, 0, 1)$.

$n = 16$, $k = 6$, $\lambda = 2$, $\mu \in \{2, 3\}$ and since

$$k(k - 1 - \lambda) = (n - k - 1)\mu$$

it follows that $\mu = 2$. Such an example of SRG with these parameters is $L_2(4)$.

Theorem 4.7. (Friendship Theorem). *If G is a simple finite graph in which every two vertices have exactly one common neighbour, there exists a (unique for $n > 3$) vertex that is adjacent to all others.*

Proof. Let G be such a graph, but assume there exists no vertex adjacent to all others. Then $n > 3$, since K_3 does not work. We'll now show this graph G is regular (of say valency k). Once we have done that, we'll know G is a SRG with parameters $(n, k, 1, 1)$. If G is complete, then any two vertices have $n - 2$ common neighbours so $n = 3$ implying G is not complete. But G is connected, by hypothesis. Also, $d(u, v) \in \{0, 1, 2\}$ for all vertices u, v. Hence, there exists two non-adjacent vertices. Let u and w be any two non-adjacent vertices and let v be their unique common neighbour. Let x and y be the common neighbours of u and v and of v and w. Let $u_1, \ldots, u_r$ be the other neighbours of u distinct from x and v, and similarly $w_1, \ldots, w_r$ other neighbours of w distinct from y and v. Next, for each other neighbour u_i of u ($u_i \neq v, x$) there exists a common neighbour u_i and w, which cannot be u, v, x or y (or otherwise get a 4-cycle and then the unique common neighbour condition is violated). Therefore, u_i is adjacent to a unique "other" neighbour w_j of w (with $w_j \neq v, y$). Then, u_i is the unique common neighbour of u and w_j. It follows that $r \leq s$, and conversely $s \leq r$ so $r = s$. Therefore,

$$\deg(u) = r + 2 = s + 2 = \deg(w).$$

SSo every two non-adjacent vertices have same degree.

Next, u, v and w be as above once more. If z is any other vertex it cannot be adjacent to both u and w, otherwise v and z would be common neighbours of v and w. Therefore either $\deg(z) = \deg(u)$ or $\deg(z) = \deg(w) = \deg(u)$, so z has same degree as u and w. Also, cannot be adjacent to every other vertex, for otherwise v would be a "host" vertex (adjacent to all others), so G would not be a counter example. Therefore, v is not adjacent to some z distinct from u, w, v and therefore

$$\deg(v) = \deg(z) = \deg(u) = \deg(w),$$

so G is regular.

Alternative proof that G is regular: Let x, y be any adjacent vertices and let z be their common neighbour. Let X, Y, Z be all other neighbours of x, y, z respectively. If $a \in X$ and $b \in B$ then

$$\deg(y) = \deg(a) = \deg(z) = \deg(b) = \deg(x),$$

so $\deg(x) = \deg(y)$ and by connectedness G is regular. Must show at least two of X, Y, Z non-empty.

Now, G is SRG with $(n, k, 1, 1)$. By property

$$k(k - \lambda - 1) = (n - k - 1)\mu,$$

it follows that $n = k^2 - k + 1$. By Rationality condition with $\Delta = (\lambda - \mu)^2 + 4(k - \mu) = 4(k - 1)$ we have that

$$f, g = \frac{1}{2}\left(k^2 - k \pm \frac{k}{\sqrt{k-1}}\right).$$

Therefore, $k-1$ is a perfect square, say $k = \ell^2 + 1$ and $\ell = \sqrt{k-1}$ divides $k = \ell^2 + 1$ so ℓ divides 1 and hence $\ell = \pm 1$. Therefore, $k = 2$ and hence $n = k^2 - k + 1 = 3$, a contradiction, and we are done. □

5. The Moore Bound

Let G be a simple finite graph with maximum degree $\Delta > 1$ and diameter D. We now ask what is the largest possible number of vertices G can have? Well, given a vertex of maximum degree, has Δ neighbours, and those neighbours have at most $\Delta - 1$ other neighbours and so on, until we get to diameter D. This gives us $\Delta(\Delta-1)^i$ possible vertices of G at depth i. That is,

$$|V(G)| \leq 1 + \Delta + \Delta(\Delta - 1) + \ldots + \Delta(\Delta-1)^{D-1} = 1 + \sum_{i=1}^{D} \Delta(\Delta-1)^{i-1},$$

which is called the *Moore Bound.* Any graph which attains this bound is called a *Moore graph.* For example, C_3 has $\Delta = 2$, $D = 1$, so achieves Moore bound $1 + 2 = 3$. Also, K_n has $\Delta = n - 1$ and $D = 1$, so $|V(K_n)| = 1 + n - 1 = n$, achieving the Moore bound. In fact, if C_n (for n odd) has $\Delta = 2$ and $D = k = \frac{n-1}{2}$ so $|V(C_n)| = 1 + \overbrace{2 + \ldots + 2}^{k} = 1 + 2k = n$.

Properties of Moore graphs: Let G be a Moore graph with max vertex Δ and diameter D such that $|V(G)| = 1 + \sum_{i=1}^{D} \Delta(\Delta-1)^{i-1}$. Then

(1) Given any vertex there must be $\Delta(\Delta-1)^{i-1}$ vertices at distance i for v;
(2) In particular, every vertex v must have Δ neighbours at distance 1, therefore G is Δ-regular.
(3) Hence number edges equals $\frac{|V(G)|}{2}\Delta > \frac{n2}{2} = n$, so G is not a tree.
(4) No two vertices at distance i from v can be adjacent if $i < D$, for otherwise Moore bound would not be attained.
(5) Similarly, no two vertices can have a common vertex at distance $i + 1$ for v.
(6) Therefore, G has no circuits of length $2i + 2$ or $2i + 1$ for $1 \leq i < D$, so Gc has girth $2D + 1$.

Another example: Peterson graph $\Delta = 3$, $D = 2$, girth $5 = 2D + 1$.

Theorem 5.1. (The Hoffman-Singleton Theorem). *If there exists a Moore graph of valence k and diameter 2, then $k \in \{2, 3, 7, 57\}$.*

Note. We do not actually know if a Moore graph that is 57-regular exists, but we do know of examples for the others. We have $k = 2$, $D = 2$ is C_5; $k = 3$, $D = 2$ is Peterson; $k = 7$, $D = 2$ is what is called the Hoffman-Singleton graph with $1 + 7 + 7(7 - 1) = 50$ vertices. Also, only Moore graphs of Diameter greater than 2 are odd length cycles $C_{2\ell+1}$, with $k = 2$ and arbitrary diameter $\ell \geq 1$.

Proof. Let G be such a Moore graph of diameter $D = 2$. By what we saw earlier, G is k-regular for some k (which we referred to as Δ earlier), and G has girth $2D + 1 = 5$, and order $1 + k + k(k - 1) = k^2 + 1$. Since G has no cycles of length 3, and no cycles of length 4, but it is connected (with diameter 2), we know that G is a SRG with parameters $\left(k^2 + 1, k, 0, 1\right)$, with $\lambda = 0$ since no triangles and $\mu = 1$ since diameter 2 and no four cycles. Hence, by TRC the numbers

$$f, g = \frac{1}{2}\left(n - 1 \pm \frac{2k + (n-1)(\lambda - \mu)}{\sqrt{(\lambda-\mu)^2 + 4(k - \mu)}}\right)$$

are non-negative integers. So,

$$f, g = \frac{1}{2}\left(k^2 \pm \frac{2k - k^2}{\sqrt{1 + 4(k-1)}}\right) = \frac{1}{2}\left(k^2 \pm \frac{(2-k)k}{\sqrt{4k-3}}\right).$$

So, $\frac{(k-2)k}{\sqrt{4k-3}}$ is an integer, with same parity as k^2 and k. If $k = 2$, this holds, so assume $k > 2$. So, $4k - 3$ is a perfect square say m^2 and then $f, g = \frac{1}{2}\left(k^2 \pm \frac{k^2-2k}{m}\right)$. In particular, m divides $k^2 - 2k$ therefore m divides

$$16k^2 - 32k = (4k)^2 - 8(4k) = \left(m^2 + 3\right)^2 - 8\left(m^2 + 3\right) = m^4 + 6m^2 + 9 - 8m^2 - 24 = m^4 - 2m^2 - 15.$$

Therefore, m divides 15 so $m = 1, 3, 5$, or possibly 15, but note $m \neq 1$ since $k = \frac{m^2+3}{4} = 1$ in this case but $k > 2$. Therefore, $m = 3, 5$ or 15 implying $k = \frac{m^2+3}{4} = 3, 7$, or 57. □

An alternate definition of Moore graphs exist, and focus on girth of the graph.

Lemma 5.2. *If C is a* girth cycle *(a cycle of length $g = \text{girth}(G)$) in a non-tree connected graph G, then $d_G(u, v) = d_C(u, v)$ for all vertices u, v of C.*

Proof. First, $d_G(u, v) \leq d_C(u, v)$, and $d_C(u, v) \leq \frac{g}{2}$ since g is of length C, and so $g \geq 2d_C(u, v)$. Let P be a path from u to v in C, and P' a path from u to v in G. If P' has length less than $\frac{g}{2}$ and P has length at most $\frac{g}{2}$, then we can create a closed walk by taking P first, then P' in reverse, which would have length less than $\frac{g}{2} + \frac{g}{2} = g$, contradicting that g is the girth of G. □

Corollary 5.3. (Moore's Inequality)*. If G is a non-tree connected graph with girth g and diameter D, then $g \leq 2D + 1$.*

Proof. Let C be girth cycle (length g) in G, and note that diameter of C is $\left\lfloor \frac{g}{2} \right\rfloor = \frac{g-1}{2}$ or $\frac{g}{2}$, depending on whether g is odd or even. Then $D = \text{Diam}(G) \geq \text{Diam}(C) \geq \frac{g-1}{2}$ and hence $g \leq 2D + 1$. □

Lemma 5.4. *Let G be non-tree connected graph diameter D. Then G has girth $2D + 1$ iff both the following conditions are satisfied:*

(1) *For every two vertices, there exists a unique shortest path between them.*
(2) *A path of length at most D between two vertices is the unique shortest path between them.*

Proof. ($\Longrightarrow$) Suppose the girth is $2D + 1$. Suppose there are two u-w paths of length at most D. Then we have a closed walk of length $2D$, a contradiction.

($\Longleftarrow$) Suppose (1) and (2) are satisfied. Let C be a girth cycle. If the length of C is $2k$, choose u, v such that $d_C(u, v) = k$, then $d_G(u, v) = d_C(u, v) = k$ (contradicting (1)). Therefore, length of C is odd, say $2k + 1$. But now suppose length $k + 1$ and k paths from u to v in cycle C, so $d_G(u, v) = k$. If $k + 1 \leq D$, contradicts (2), so $k + 1 > D$ giving us $k \geq D$. So, $2k + 1 \geq 2D + 1$ implying girth is $2D + 1$. □

Theorem 5.5. *Let G be a connected non-tree finite simple graph with diameter D and girth $2D + 1$. Then G is regular, and has order $1 + \sum_{1 \leq i \leq D} \Delta(\Delta - 1)^{i-1}$, where Δ is the valency of G.*

Proof. Suppose P is a v-w path of length D. Any neighbour u of v that does not lie in path P has $d_G(u, w) \leq D$ but $d_G(u, w) \geq D$ because we'd get a closed walk of length less than $1 + D + D = 2D + 1$ which is the girth (a contradiction. Now, z any unique neighbour of w in shortest path from u to w. This shows $\deg(v) \leq \deg(w)$, and by symmetry $\deg(w) \leq \deg(v)$ so $\deg(v) = \deg(w)$.

Now, vertices on C have same valency (just consider taking opposite vertices, i.e., distance k) Then given any vertex u in G which has distance k from some v_0 in C, taking w which has distance $D - k$ from v_0 in C, $\deg(u) = \deg(w) = \deg(v_i)$ for all v_i in C. Therefore, G is regular.

If valency of G is Δ, then $|V(G)| \geq 1 + \Delta + \Delta(\Delta - 1) + \ldots + \Delta(\Delta - 1)^{D-1}$ by a kind of 'dual' of the "Moore bound" argument. But conversely, $|V(G)| \leq RHS$ because diameter is D. □

Theorem 5.6. *Let G be a connected non-tree simple graph with order n, diameter D and maximum valence Δ. Then $n = 1 + \sum_{1 \leq i \leq D} \Delta(\Delta - 1)^{i-1}$ if and only if G has girth $2D + 1$, and then in each case, G is regular with valency Δ.*

6. Extremal graph theory

The maximum number of edges which contains no cycles is $n-1$. The maximum number of edges containing no odd cycles is $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$ (also is the maximum number of edges containing no three-cycles).

The Turan graph $T(n,r)$ is complete multi-partite graph of order n having r parts which are edgeless graphs with size $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$.

Take $n = kr + L$, then $r - L$ parts of size $\lfloor \frac{n}{r} \rfloor$, L of parts of size $\lceil \frac{n}{r} \rceil$. Denote $t(n,r)$ by the number of edges of $T(n,r)$.

Theorem 6.1. (Turan's Theorem). *Let G be a simple graph on n vertices and m edges, and let r be any positive integer. If G has no complete subgraph of order $r+1$, then $m \leq t(n,r)$, and $m = t(n,r)$ iff G is isomorphic to the graph $T(n,r)$.*

Note. $r = 2$: if G contains $C_3 = K_3$, $m \leq t(n,2) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. If $n \leq r$, then $t(t,r) = \binom{n}{2}$. If $n > r$, then $t(n,r) = t(n-r,r) + \binom{r}{2} + (n-r)(r-1)$.

Proof. By induction. Base case is $n \leq r$. Then $T(n,r) = K_n$, so the statement is true (including statement about equality). Inductive step: Let G be a graph with no K_{r+1} subgraph, and with maximum number of possible edges. If G contains no K_r, then $G + e$ contains no K_{r+1} for any edge $e \notin E(G)$, a contradiction. Let X be the vertex set for $K_r \leq G$. Let $Y = V(G) \setminus X$. Let H be the induced subgraph of $G[Y]$. H contains no K_{r+1}, implying at most $t(n-r,r)$ edges in H. There are $\binom{r}{2}$ edges inside the set X. If some vertex v of H has r neighbours in X, then G would contain K_{r+1} subgraph. Hence, there are at most $(n-r)(r-1)$ edges from Y to X. Therefore, this gives us that

$$m = |E(G)| \leq t(n-r,r) + \binom{r}{2} + (n-r)(r-1) = t(n,r).$$

If equality holds, then $|E(H)| = t(n-r,r)$ and there are exactly $(n-r)(r-1)$ edges from Y to X. By induction, $H \cong T(n-r,r)$. Also, $(n-r)(r-1)$ makes every vertex in Y is adjacent to all but one vertex in X. Also, if u, v are in different parts of $T(n-r,r)$, then they miss different vertices of X in their neighbourhoods (since otherwise K_{r+1}). This finishes the proof, so $G \cong T(n,r)$, because every $x \in X$ can be added to the unique part which does not contain x as a neighbour. □

7. Ramsey Theory

Theorem 7.1. (Ramsey's Theorem). *For all positive integers s and t, there exists an integer N_0 with property that if $N \geq N_0$, then every partition of the edges of the complete graph K_N into red and blue edges has a red-edge subgraph isomorphic to K_s, or a blue-edge subgraph isomorphic to K_t. Moreover,*

$$N_0 \leq \binom{s+t-2}{s-1} = \binom{s+t-2}{t-1}.$$

Proof. By induction. Base case: $s = 1$ or $t = 1$ works. Induction step: assume we found $N_0(s-1,t)$ such that colouring edges of $K_{N_0(s-1,t)}$ gives red K_{s-1} or blue K_t, and similarly $N_0(s,t-1)$ (red K_s or blue K_{t-1}). Let $N_0 = N_0(s-1,t) + N_0(s,t-1)$. We let $v \in K_{N_0}$, and assume edges of K_{N_0} are coloured red and blue. v is incident to either $N_0(s-1,t)$ red edges or to $N_0(s,t-1)$ blue edges (consider counting argument, there are $N_0(s-1,t) + N_0(s,t-1) - 1$ other vertices besides v). If v incident to $N_0(s-1,t)$ red edges, let X_{red} be the set of neighbours of v connected by red edges. Then $|X_{red}| \geq N_0(s-1,t)$ implies either find a K_{s-1} vertices in X_{red} and all edges red, or a K_t with vertices in X_{red} and all edges blue.

In the first case, we can add v to K_{s-1} to obtain red K_s. If v incident to $N_0(s,t-1)$ blue edges, swap roles of s and t. Hence,

$$N_0(s,t) \leq N_0(s-1,t) + N_0(s,t-1) \leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}.$$

□

Define $R(s,t)$ to be minimum positive integer which satisfies the above. Observe

$$R(s,s) \leq \binom{2s-2}{s-1} \leq \binom{2s}{s} \approx \frac{4^s}{\sqrt{\pi s}}$$

and $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Theorem 7.2. *$R(t,t) \geq 2^{\frac{t}{2}}$ for every $t \geq 3$.*

Proof. Pick a colouring at random on graph with $n = 2^{\frac{t}{2}}$ vertices, i.e., each edge is red with probability $\frac{1}{2}$ and blue with probability $\frac{1}{2}$. For a set X of t vertices, the probability of a monochromatic K_t with vertex set X is

$$2 \times \left(\frac{1}{2}\right)^{\binom{t}{2}},$$

since we add probability all blue and all red edges. The probability that there is a monochromatic K_t is at most

$$\sum_{X \subseteq V : |X| = t} \frac{2}{2^{\frac{t^2-t}{2}}} = \binom{n}{t} \frac{2}{2^{\frac{t^2-t}{2}}} \leq \frac{n^t}{t!} \frac{2}{2^{\frac{t^2-t}{2}}} = \frac{2^{1+\frac{t}{2}}}{t!} < 1.$$

Hence, the probability of no monochromatic K_t is greater than 0, there exists red-blue colouring of K_n with no monochromatic K_t. □

No bound $\left(\sqrt{2}+\epsilon\right)^t$ is known for any $\epsilon > 0$.

Theorem 7.3. *If for $N \in \mathbb{N}$ there is $p \in [0,1]$ such that $\binom{N}{s} p^{\binom{s}{2}} + \binom{N}{t} (1-p)^{\binom{t}{2}} < 1$, then $R(s,t) > N$.*

Proof. Pick colouring: colour red prob p and blue prob $1-p$. Probability of a fixed set on s vertices with only red edges is $p^{\frac{s^2-s}{2}}$ and probability fixed set on t vertices with only blue edges is $(1-p)^{\frac{t^2-t}{2}}$. Probability there is a red K_s or blue K_t at most sum over all of them, which is less than 1 by assumption. □

Theorem 7.4. *$R(3,t) > \frac{t^{5/4}}{2\log(t)} =: N$.*

Proof. Let $p = \frac{1}{N}$. Then using $(1+x) \leq e^x$, we obtain

$$\begin{aligned}
\binom{N}{3} p^{\binom{3}{2}} + \binom{N}{t} (1-p)^{\binom{t}{2}} &\leq \frac{N^3}{3!}\frac{1}{N^3} + \frac{N^t}{t!} e^{-p\frac{t^2-t}{2}} \\
&\leq \frac{1}{6} + \frac{t^{\frac{5}{4}t}}{2^t t!} e^{-2\frac{\log(t)}{t^{\frac{5}{4}}}\frac{t^2-t}{2}} \\
&\leq \frac{1}{6} + \frac{t^{\frac{5}{4}t}}{2^t \left(\frac{t}{2}\right)^{\frac{t}{2}}} t^{\frac{-t^2+t}{t^{\frac{5}{4}}}} \\
&\leq \frac{1}{6} + \frac{t^{\frac{3}{4}t}}{\sqrt{2}^t} t^{-t^{\frac{3}{4}}+1} \leq \frac{1}{6} + \frac{t}{\sqrt{2}^t} < 1.
\end{aligned}$$

□

Note. $R(3,t) \approx c\frac{t^2}{\log(t)}$.

Theorem 7.5. *Let $x_1, x_2, x_3, \dots, x_n$ be natural numbers. Then there exists a subset of size $\geq \frac{n}{3}$ of $X = \{x_1, \dots, x_n\}$ such that $x_i + x_j \neq x_k$ for any triple i, j, k.*

Proof. Let $p = 3k+2$ be a prime such that $p > \max_{1 \leq i \leq n} x_i$. The set $K = \{k+1, k+2, \dots, 2k+1\}$ satisfies $a + b \neq c$ for any a, b, c in the set K. Pick $y \in \mathbb{Z}_p \setminus \{0\}$ at random. Consider $yX = \{yx_i \in \mathbb{Z}_p \mid 1 \leq i \leq n\}$. Consider $\sigma_i = \begin{cases} 1 & \text{if } yx_i \in K; \\ 0 & \text{otherwise;} \end{cases}$

$$\mathbb{P}(\sigma_i = 1) = \frac{|K|}{p-1} = \frac{k+1}{3k+1} > \frac{1}{3}.$$

For all $a, b \in \mathbb{Z}_p \setminus \{0\}$ there exists unique x such that $ax = b \mod p$. Now,

$$\mathbb{E}(\#i \text{ s.t. } yx_i \in K) = \mathbb{E}\left(\sum_{i=1}^{n} \sigma_i\right) = \sum_{i=1}^{n} \mathbb{E}(\sigma_i) > \frac{n}{3}.$$

Therefore, there exists y such that $|yX \cap K| > \frac{n}{3}$. Let $X' \subseteq X$ be the set $\{x \in X \mid yx \in K\}$. If $x_i, x_j, x_k \in X'$ such that $x_i + x_j = x_k$, then $yx_i + yx_j = yx_k$, but $yx_i + yx_j \not\equiv yx_k \mod p$, so we are done. ☐

8. Group automorphisms

A group action $\varphi : G \to \mathrm{Sym}(X)$ is *faithful* if it has trivial kernel, i.e., $G \cong \mathrm{im}(\varphi) \leq \mathrm{Sym}(X)$. Note $\mathrm{Sym}(X)$ is the symmetry group on X (all bijections from X to itself under composition). Now, G acting on X, $g \in G$ and $x \in X$ we write gx instead of $(\varphi(x))(x)$. Now, the *orbit* is $Gx = \{gx \mid g \in G\}$ and the *stabilizer* is $G_x = \{g \in G \mid gx = x\}$. Observe $G_x \cong \mathrm{Sym}(X \setminus \{x\})$. Now, the action is *transitive* if only one orbit, i.e., $Gx = X$ for each $x \in X$.

Theorem 8.1. (Orbit-Stabilizer Theorem). *Let G be a group acting on a finite set. Then $|G| = |Gx||G_x|$.*

Proof. Let $x \in X$. Take $Y = \{(g, y) \mid g \in G, gx = y\}$. Observe

$$|Y| = \sum_{g \in G} |\{y \mid y = gx\}| = |G|,$$

and

$$|Y| = \sum_{y \in Gx} |\{g \in G \mid gx = y\}|,$$

for each $y \in Gx$ choose $h_y \in G$ such that $h_y x = y$. Then

$$\{g \in G \mid gx = y\} = \left\{g \in G \;\middle|\; h_y^{-1} gx = x\right\} = h_y \cdot \{h \in G \mid hx = x\} = h_y G_x.$$

Hence,

$$|Y| = \sum_{y \in Gx} |G_x| = |Gx||G_x|,$$

and thus $|G| = |Gx||G_x|$. ☐

An *automorphism* of a graph G is a permutation of the vertices of G that preserves adjacency. The automorphisms of G form a group under composition called the *automorphism group* of G and denoted by $\mathrm{Aut}(G) \leq \mathrm{Sym}(V)$.

Example 8.2. Consider the cyclic graph $G = C_n$ on n vertices. Obvious automorphisms include n rotations and n reflections (and any combination of these). Hence, the dihedral group D_n is isomorphic to a subgroup of $\mathrm{Aut}(G)$. We claim D_n is isomorphic to $\mathrm{Aut}(G)$. Since $|D_n| = 2n$, suffices to show $|\mathrm{Aut}(G)| = 2n$.

Write $\Gamma = \mathrm{Aut}\,(G)$. Now, fix vertex v of G and let u be adjacent to v. By the Orbit-Stabilizer theorem,

$$|\Gamma| = |\Gamma v||\Gamma_v|.$$

Observe that the group action is transitive, implying $|\Gamma v| = n$. As Γ_v is a subgroup of Γ, we may apply the Orbit-Stabilizer once again to yield

$$|\Gamma_v| = |\Gamma_v u||(\Gamma_v)_u| = |\Gamma_v u||\Gamma_{v,u}|.$$

Now, if $g \in \Gamma_v$ then $gu \sim gv = v$, so $|\Gamma_v u| \leq \deg v = 2$. On the other hand, reflection shows that $|\Gamma_v u| \geq 2$, so $|\Gamma_v u| = 2$. For $\Gamma_{v,u}$, note that if $g \in \Gamma_{v,u}$ and $w \neq u$ is other neighbour of v, then $gw \sim gv = v$, but $gw \neq u$ because $gu = u$. So, $gw = w$. By induction, $gx = x$ for all vertices x. Hence, $g = \mathrm{id}_V$, so $|\Gamma_{v,u}| = 1$. Thus, $|\Gamma| = 2n$, so $\Gamma \cong D_n$.

Example 8.3. Consider $J(n, k, r)$, where vertex set is set of k-element subsets of $\{1, \ldots, n\}$ and have edges from A to B if $|A \cap B| = r$. Peterson graph is $J(5, 2, 0)$. Take $G = J(6, 4, 3)$. Observe $S_6 = \mathrm{Sym}\,(\{1, \ldots, 6\}) \leq \mathrm{Aut}\,(G)$, where $g\{a, b, c, d\} = \{ga, gb, gc, gd\}$. If $A \cap B$ has 3 elements, then $|gA \cap gB| = 3$, so defines automorphism for each $g \in S_6$. Let $\Gamma = \mathrm{Aut}\,(G)$, $u = \{1, 2, 3, 4\}$. So, $|\Gamma| = |\Gamma u||\Gamma_u|$ where S_6 acts transitively implying Γ acts transitively and so $|\Gamma u| = |V| = \binom{6}{4} = 15$. Next, let v be any neighbour of u (e.g., we take $v = \{2, 3, 4, 5\}$). Then

$$|\Gamma_u| = |\Gamma_u v||\Gamma_{u,v}|.$$

We have $\deg\,(u) = \binom{4}{3}\binom{2}{1} = 8$. For $v' = \{1, 2, 4, 5\}$, $v = \{2, 3, 4, 5\}$, we get $g(1) = 3$, $g(3) = 1$, $g(i) = i$ for all $i \neq 1, 3$ has $g \in \Gamma_v$, $gv' = v$. Similarly, v can be mapped to all $x \sim v$ implying $|\Gamma_u v| = 8$. Now,

$$|\Gamma_{u,v}| = |\Gamma_{u,v} w||\Gamma_{u,v,w}|$$

for $v \in N(v) \setminus N(u)$ implying w contains two elements of $\{2, 3, 4\}$ and 2 of $\{5, 6\}$. $(S_6)_{u,v}$ acts transitively on $N(v) \setminus N(u)$, so $|N(v) \setminus N(u)| = |\Gamma_{u,v} w| = 3$. Finally,

$$|\Gamma_{u,v,w}| = |\Gamma_{u,v,w} z||\Gamma_{u,v,w,z}|$$

where $z \in N(w) \setminus (N(u) \cup N(v))$ and $(S_6)_{u,v,w}$ acts transitively and has size 2. So,

$$|\Gamma| = 15 \times 8 \times 3 \times 2 \times 1 = 6!,$$

giving us $\Gamma \cong S_6$.

Induced actions on edges refer to $V(L(G))$. Also have induced actions on cycles and spanning trees.

Theorem 8.4. *If G is a connected simple finite graph with at least three vertices, then the action of* $\mathrm{Aut}\,(G)$ *on the edges of G is faithful (i.e., only element fixing everything is the identity).*

Proof. Assume that there exists $g \in \mathrm{Aut}\,(G)$ which fixes all edges, but not all vertices. Let $v \in V$ such that $gv \neq v$, let e be an edge incident to v, say $e = \{v, w\}$. Since $g(e) = e$, we know that $g(v) \in \{v, w\}$ implying $g(v) = w$ and $g(w) = v$. At least one of v, w must have another neighbour. Without loss of generality, w has a neighbour $x \neq v$. Let $f = \{w, x\}$. By same argument as before, $g(w) \in \{x, w\}$. But $g(w) = v$, so would imply $v \in \{x, w\}$, a contradiction. □

Corollary 8.5. $\mathrm{Aut}\,(G)$ *is isomorphic to a subgroup of* $\mathrm{Aut}\,(L(G))$, *provided* $|V| \geq 3$ *and G is finite simple connected.*

Theorem 8.6. *If G is a connected simple graph with at least five vertices, then* $\mathrm{Aut}\,(L(G))$ *is isomorphic to* $\mathrm{Aut}\,(G)$.

Proof. For $v \in V$, let $E(v)$ denote the set of edges incident to v. Observe that if $|E(v) \cap E(w)| \geq 2$, then $v = w$ (otherwise there would be 2 edges connecting v to w, contradicting G simple). Need to show for $g \in \text{Aut}\,(L(G))$ there is $g^* \in \text{Aut}\,(G)$ such that $g^*(e) = g(e)$ for all edges.

Claim: Let $g \in \text{Aut}\,(G)$, $v \in V(G)$, then there is $v^* \in V(G)$ such that $g(E(v)) \subseteq E(v^*)$ (where v^* is our candidate choice for $g^*(v)$). Assume no such v^* exists. The only way edges form a clique in $L(G)$, but do not share a vertex is for them to form a triangle in G. This implies $\deg(v) = 3$ and $g(E(v))$ to be edges of a triangle. Let $\{e, e', e''\} = E(v)$. Since G has ≥ 5 vertices, there is a vertex w such that $d(v, w) = 2$. Let u be the common vertex of v and w. Let $f = \{u, w\}$. Without loss of generality, $\{u, v\} = e$. e and f are adjacent in $L(G)$, implying $g(e)$ and $g(f)$ are adjacent in $L(G)$, implying $g(f)$ contains one of the vertices in the triangle spanned by $g(e)$, $g(e')$ and $g(e'')$. Hence, $g(f)$ is adjacent to one of $g(e')$ or $g(e'')$ in $L(G)$. It follows f is adjacent to e' or e'', that is, intersects one of them. That is, intersect in w. But then w is either v or a neighbour of v, contradiction.

Same for g^{-1} and v^* instead of g and v, implying v^{**} has $g^{-1}(E(v^*)) \subseteq E(v^{**})$. Observe $E(v) = g^{-1}(g(E(v))) \subseteq g^{-1}(E(v^*)) \subseteq E(v^{**})$. If $\deg(v) \geq 2$, then $E(v) \cap E(v^{**}) = E(v)$ has cardinality at least 2, implying $v = v^{**}$ by previous observation. So, $g^{-1}(g(E(v))) = g^{-1}(E(v^*))$ implying $g(E(v)) = E(v^*)$. If $\deg(v) = 1$, then v has a neighbour w with $\deg(w) \geq 2$. Therefore, $g(E(w)) = E(w^*)$ for some $w^* \in V(G)$. Let $f = \{v, w\}$, implying $g(f) \in E(w^*)$. Number of edges adjacent to f in $L(G)$ is the same as $g(f)$, and also equal to $\deg(v) + \deg(w) - 2 = \deg(w^*) + \deg(v^*) - 2$ (where v^* other endpoint of $g(f)$). This means that v^* has degree one and thus $g(E(v)) = E(v^*)$ (where $E(v) = \{f\}$ and $E(v^*) = \{g(f)\}$ in this case).

Observation: if $v \neq w$ then $E(v) \neq E(w)$ where G connected on ≥ 5 vertices. This implies that for each v there is a unique v^* such that $g(E(v)) = E(v^*)$. Define $g^* : V(G) \to V(G)$. Then by previous argument, well-defined and one-to-one (and by finiteness of $V(G)$, also onto). Suppose $v \sim u$. This happens iff $E(v) \cap E(u) \neq \varnothing$ iff $E(v^*) \cap E(u^*) \neq \varnothing$ iff $v^* \sim u^*$. So preserves adjacency implies g^* is an automorphism. □

Definition 8.7. A graph is called

- *vertex-transitive* if its automorphism group acts transitively (on vertices)
- *edge-transitive* if the induced action on edges is transitive
- *arc-transitive* if induced action on arcs (ordered walks length 1) is transitive
- *s-arc-transitive* if induced action on s-arcs (ordered walks length s) is transitive.

s-arc transitive does not necessarily imply arc-transitive: consider tree on four vertices (in a line). Then G is 3-arc-transitive, but not arc-transitive.

8.1. **Cayley graphs.** Let G be a group and let S be an identity-free inverse closed subset of G. The (left) *Cayley graph* denoted $Cay(G, S)$ on G with *connection set* S is the graph with vertex set G and edge set $\{\{g, gs\} \mid g \in G, s \in S\}$. For example, $G = C_7 = (\mathbb{Z}_7, +)$ take $S = \{1, 6, 3, 4\}$.

If identity in S, then $Cay(G, S)$ is not simple (loop at every vertex). S generates G iff $Cay(G, S)$ is connected (sometimes included as assumption). Since S is inverse closed: there is an edge (by def) from gs to $gss^{-1} = g$. If inverse closed is not assumed, then it makes sense to view Cayley graphs as directed.

G acts on itself by group operation: $g(h) = g \circ h$. The action of a group element g on $h \in G$ is given by "left multiplication".

Definition 8.8. A group G acting on a set X is *semi-regular* if $G_x = \{1\}$ for all $x \in X$, and *regular* if semi-regular and transitive. Semi-regular implies at most one element mapping x to y for each pair x, y. Recall transitive means every element can be mapped to any other.

Action of G on itself is regular: semiregular if exists x s.t. $g(x) = x$. Then by def $gx = x$ implies g is identity. Transitive: say $x, y \in G$, then $g = yx^{-1}$ has $g(x) = yx^{-1}x = y$.

Theorem 8.9. *Let Γ be a simple graph and let G be a group. Then Γ is isomorphic to a Cayley graph on G iff* $\mathrm{Aut}(\Gamma)$ *contains a regular subgroup isomorphic to G.*

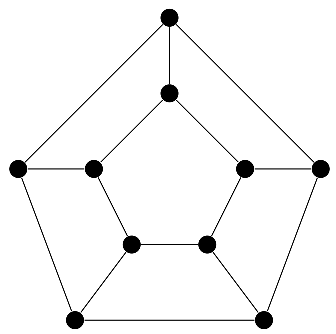
Proof. ($\Longrightarrow$). Let $\Gamma = \mathrm{Cay}(G, S)$ for some $S \subseteq G$. The vertices of Γ connected to elements of G, each $g \in G$ induces a permutation on G by $g(x) = g \cdot x$. This is an automorphism: $x \sim y$ iff $y = x \cdot s$ for some $s \in S$ iff $g \cdot y = g \cdot x \cdot s$ iff $g(x) \sim g(y)$.

Let G be a subgroup of $\mathrm{Aut}(\Gamma)$ acting regularly on V. Pick some $v \in V$, for every $w \in V$, label w be the unique $g_w \in G$ such that $g_w v = w$.

Let S be the set of labels of neighbours of v. Let $\{x, y\} \in E$, need to show that $g_y = g_x \cdot s$ for some $s \in S$. Let g_x be the label of x, then $\left\{g_x^{-1}x, g_x^{-1}y\right\} \in E$ (where $g_x^{-1}x = v$ and $g_x^{-1}y \in N(v)$). Then $g_x^{-1}g_y \in S$ implying $g_y = g_x \cdot s$ for some $s \in S$ (conversely holds as well). □

Corollary 8.10. *Cayley graphs are vertex-transitive.*

Example 8.11. D_5 (dihedral group acting on 5 elements) acts regularly. $\mathbb{Z}_5 \times \mathbb{Z}_2$ also acts regularly (these are the only groups of order 10). Both on graph below.



K_n is vertex transitive, and edge transitive because if $\{x, y\}$, $\{x', y'\} \in E$, then the permutation $(x, x')(y, y')$ or similar sends x to x' and y to y'. arc-transitive by similar argument; also 2-arc transitive since we fix middle vertex, send it somewhere, then permute two end points. 3-arc transitive iff $n \leq 3$ (consider K_3). C_n is s-arc transitive for every s.

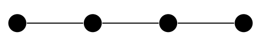
Theorem 8.12. *If Γ is a finite graph of minimal valency of at least* 3*, then there is a largest s such that Γ is s-arc-transitive.*

Proof. When constructing an s-arc, we have at least 2 possibilities to pick the next vertex in each step. This implies at least 2^s different s-arcs. But $\mathrm{Aut}(\Gamma)$ is finite, so exists s such that $2^s > |\mathrm{Aut}(\Gamma)|$. If there are more s-arcs then automorphisms, then $\mathrm{Aut}(\Gamma)$ cannot be transitive.

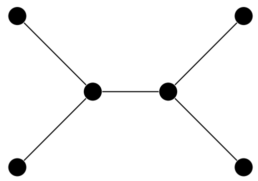
A second proof is as follows: minimum degree at least 3, so implies there is a cycle. Let C be a shortest cycle. There is a $|C|$-arc forming a cycle, and a $|C|$-arc which is not a cycle. So we actually obtain an upper bound for s, which is $|C|$. □

Theorem 8.13. *If $s \geq 1$ and Γ is s-arc-transitive and has minimal valency at least* 2*, then G is $(s-1)$-arc-transitive.*

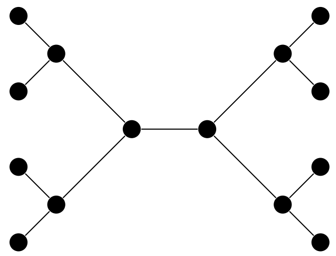
Example 8.14. The following are 3-arc-transitive but not 1- or 2- arc transitive.



and



The following is 5-arc transitive, but not s-arc-transitive for every $s < 5$.



Theorem 8.15. *If $s \geq 1$ and Γ is s-arc-transitive and has minimal valency at least 2, then G is $(s-1)$-arc-transitive.*

Proof. Let $v_0, v_1, \ldots, v_{s-1}$ and $u_0, \ldots, u_{s-1}$ be $(s-1)$-arcs. Since $\deg(v_{s-1}) \geq 2$, there is $v_s \sim v_{s-1}$ and $v_s \neq v_{s-2}$ implying $v_0, v_1, \ldots, v_{s-1}, v_s$ is an s-arc. Similarly for $u_0, \ldots, u_{s-1}, u_s$. So can send $v_0, \ldots, v_s$ to $u_0, \ldots, u_s$ implying that we send $v_0, \ldots, v_{s-1}$ to $u_0, \ldots, u_{s-1}$. □

Theorem 8.16. *Let $s \geq 0$ and Γ be an s-arc-transitive graph and let $(v_0, \ldots, v_s)$ be an s-arc of Γ. Then Γ is $(s+1)$-arc-transitive if and only if $\mathrm{Aut}(\Gamma)_{(v_0,\ldots,v_s)}$ acts transitively on $N(v_s) \setminus \{v_{s-1}\}$.*

Proof. ($\Longrightarrow$) Assume $\mathrm{Aut}(\Gamma)$ is $(s+1)$-arc-transitive. Let $v_{s+1}, v'_{s+1} \in N(v_s) \setminus \{v_{s-1}\}$. Then $v_0, \ldots, v_s, v_{s+1}$ and $v_0, \ldots, v_s, v'_{s+1}$ are $(s+1)$-arcs, so there exists $g \in \mathrm{Aut}(\Gamma)$ such that $g(v_i) = v_i$ for all $0 \leq i \leq s$ and $g(v_{s+1}) = v'_{s+1}$ implying $g \in (\mathrm{Aut}(\Gamma))_{(v_0,v_1,\ldots,v_s)}$. This is for any pair v_{s+1}, v'_{s+1}, so implies $\mathrm{Aut}(\Gamma)_{(v_0,\ldots,v_s)}$ acts transitively on $N(v_s) \setminus \{v_{s-1}\}$.

($\Longleftarrow$) Let $v_0, \ldots, v_{s+1}$ be an $(s+1)$-arc. Let $u_0, \ldots, u_{s+1}$ be any other $(s+1)$-arc. Need to show that there exists $g \in \mathrm{Aut}(\Gamma)$ such that $g(v_i) = u_i$ for all $0 \leq i \leq s+1$. By s-arc-transitivity: exists $h \in \mathrm{Aut}(\Gamma)$ such that $h(u_i) = v_i$ for all $0 \leq i \leq s$. Let $v'_{s+1} = h(v_{s+1})$. We know $v'_{s+1} \in N(v_s) \setminus \{v_{s-1}\}$. There exists $k \in \mathrm{Aut}(\Gamma)_{(v_0,\ldots,v_s)}$ such that $k\big(v'_{s+1}\big) = v_{s+1}$. Taking $g := k \circ h$ satisfies $g(u_i) = k(h(u_i)) = v_i$ for all $0 \leq i \leq s+1$. □

Example 8.17. Take $\Gamma = K_5$. Fixing two vertices x and y, we have $\mathrm{Aut}(\Gamma)_{(x,y)} \cong S_3$ acts transitively on remaining vertices. Fixing three vertices x, y, z, can only permute remaining two (so isomorphic to S_2). Then not transitive on $N(z) \setminus \setminus \{y\} = \{x, u_1, u_2\}$ implying not 3-arc-transitive.

A graph Γ is called

- *arc regular* if the induced action of $\mathrm{Aut}(\Gamma)$ on arcs is regular (i.e., transitive and semi-regular, where semi-regular means that $\mathrm{Aut}(\Gamma)_{(v_0,v_1)}$ is trivial).
- *s-arc regular* if the induced action on s-arcs is regular.

Example 8.18. $\mathrm{Aut}(K_4)$ is 2-arc regular. It is 2-arc transitive by previous theorem, and for any 2-arc (v_0, v_1, v_2) the stabilizer is the identity (since we must fix remaining vertex as well). That is, semi-regular.

Theorem 8.19. (Tutte). *Let* Γ *be a connected* 3*-regular arc-transitive graph. Then there exists some* $s \in \mathbb{N}$ *such that* Γ *is* s*-arc regular.*

Proof. Let s be maximal such that Γ is s-arc-transitive. Let $(v_0, \dots, v_s)$ be an s-arc. v_s has three neighbours, since 3-regular, so $N(v_s) = \{v_{s-1}, x, y\}$. Since $\mathrm{Aut}(\Gamma)$ is not $(s+1)$-arc-transitive, we know that for every $g \in \mathrm{Aut}(\Gamma)_{(v_0,\dots,v_s)}$ fixes x and y (and consequently fixes $N(x) \setminus \{v_s\}$). Hence, $\mathrm{Aut}(\Gamma)_{(v_1,\dots,v_s,x)} \leq \mathrm{Aut}(\Gamma)_{(v_0,\dots,v_s)}$. Therefore, every $g \in \mathrm{Aut}(\Gamma)_{(v_0,\dots,v_s)}$ fixes vertex at distance 2 from v_s. Connectedness and induction implies every $g \in \mathrm{Aut}(\Gamma)_{(v_0,\dots,v_s)}$ fixes every vertex at distance at most $\mathrm{diam}(\Gamma)$ from v_s. Therefore, $\mathrm{Aut}(\Gamma)_{(v_0,\dots,v_s)}$ is the identity. □

9. Generating functions

A *formal power series* in variable z is an expression of the form $\sum_{n=0}^{\infty} a_n z^n$, where $a_n \in \mathbb{R}$.

Example 9.1. $a_n = 1$, $A(z) = \sum_{n=0}^{\infty} 1 \cdot z^n = \frac{1}{1-z}$ for z small.
$a_n = 2^n$, $A(z) = \sum_{n=0}^{\infty} (2z)^n = \frac{1}{1-2z}$.
$a_n = \frac{1}{n!}$, $A(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$.

The *generating function* of a sequence (a_n) is the formal power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$, where a_n is the *coefficient of* z^n on $A(z)$, denoted by $[z^n]A(z) = a_n$.

Example 9.2. $a_0 = a_1 = 2$, $a_n = 2a_{n-1} + 3a_{n-2}$; want to make this explicit. We have that

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n z^n &= 2 + 2z + \left(\sum_{n=2}^{\infty} (2a_{n-1} + 3a_{n-2}) z^n \right) \\
&= 2 + 2z + \left(2z \sum_{n=2}^{\infty} a_{n-1} z^{n-1} \right) + \left(3z^2 \sum_{n=2}^{\infty} a_{n-2} z^{n-2} \right) \\
&= 2 - 2z + \left(2z \sum_{n=0}^{\infty} a_n z^n \right) + 3z^2 \sum_{n=0}^{\infty} a_n z^n \\
&= 2 - 2z + 2zA(z) + 3z^2 A(z).
\end{aligned}$$

Hence, $A(z) = \frac{2z-2}{3z^2+2z-1} = \frac{2z-2}{(1+z)(3z-1)} = \frac{1}{1+z} + \frac{1}{1-3z}$. Consequently, $A(z) = \sum_{n=0}^{\infty} (-z)^n + \sum_{n=0}^{\infty} 3^n z^n$, giving us $a_n = (-1)^n + 3^n$.

Recall Cauchy product formula: $(\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n) = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_n b_{n-k})$.

Ordered rooted binary trees: every node is either a leaf, or has a left child or a right child or both. How many different ones are there, given number of vertices? Let t_n be the number of ordered rooted trees with n vertices. Then $t_0 = 1$ (empty tree), $t_1 = 1$, $t_2 = 2$, $t_3 = 5$.

A non-empty ordered rooted tree consists of:

- a root
- a left child ordered rooted tree (could be empty)
- a right child ordered rooted tree (could be empty)

If a tree has n vertices, then the number of vertices in the left child tree plus the number of vertices in right subtree equals $n-1$. Therefore, $t_{n+1} = \sum_{k=0}^{n} t_k t_{n-k}$

for $n \geq 0$. Hence

$$\begin{aligned}\sum_{n=0}^{\infty} t_n z^n &= 1 + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} t_k t_{n-k} z^{n+1}\right)\\ &= 1 + z\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} t_k z^k t_{n-k} z^{n-k}\right)\\ &= 1 + z\left(\sum_{n=0}^{\infty} t_n z^n\right)\left(\sum_{n=0}^{\infty} t_n z^n\right).\end{aligned}$$

Therefore, $T(z) = 1 + z(T(z))^2$. Consequently,

$$T(z) = \frac{1 \pm \sqrt{1-4z}}{2z},$$

and

$$\sqrt{1-x} = 1 - \sum_{n=1}^{\infty} \frac{1 \cdot 2 \cdot 5 \cdot \ldots \cdot (2n-3)}{2^n n!} x^n$$

implies

$$T(z) = \frac{1}{2z}\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{2^n n!} 4^n z^n = \frac{1}{z}\sum_{n=1}^{\infty}\frac{1}{n}\binom{2n-2}{n-1} z^n = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n} z^n.$$

Thus, $t_n = \frac{1}{n+1}\binom{2n}{n}$.

Example 9.3. $\sum_{n=0}^{\infty} n! z^n$ is a formal power series but does not converge for $z \neq 0$.

For generating functions, we have

$$\begin{aligned}&C(z) = A(z) + B(z) \iff c_n = a_n + b_n \text{ (Addition)}\\ &C(z) = sA(z) \iff c_n = s a_n \text{ (Scaling)}\\ &C(z) = zA(z) \iff c_0 = 0 \text{ and } c_n = a_{n-1} \text{ for } n > 0 \text{ (Right shift)}\\ &C(z) = A(z)B(z) \iff c_n = \sum_{k=0}^{n} a_k b_{n-k} \text{ (Convolution)}\\ &C(z) = A'(z) \iff c_n = (n+1)a_{n+1},\end{aligned}$$

and we call $A'(z)$ *formal derivative.*

Example 9.4. $(\sum_{n=0}^{\infty} z^n)(\sum_{n=0}^{\infty} n! z^n) = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} k!) z^n$.
$A(z)\big(sz^0 + \sum_{n=1}^{\infty} 0z^n\big) = sA(z)$. Also, $A(z)\big(0z^0 + 1z^1 + \sum_{n=2}^{\infty} 0z^n\big) = zA(z)$.

Note. Formal power series with coefficients in $\mathbb{R}$, they form a unitary ring w.r.t. $+$, $\cdot$. The (multiplicative) identity element in this ring is $I(z) = 1 \cdot z^0 + \sum_{n=0}^{\infty} 0z^n$.

Lemma 9.5. *A formal power series $A(z)$ has a multiplicative inverse if and only if $[z^0]A(z) \neq 0$.*

Proof. ($\Longleftarrow$) Let $a_n := [z^n]A(z)$ and assume $a_0 \neq 0$. Let $b_0 = \frac{1}{a_0}$, $b_0 = -\frac{1}{a_0}\sum_{k=1}^{n} a_k b_{n-k}$ for $n \geq 1$. Let $B(z) = \sum_{n=0}^{\infty} b_n z^n$. Claim: $B(z) \cdot A(z) = I(z)$.

$$\begin{aligned}[z^n](A(z)B(z)) &= \sum_{k=0}^{n} a_k b_{n-k}\\ &= a_0 b_n + \sum_{k=1}^{n} a_k b_{n-k}\\ &= \begin{cases} a_0 b_0 = 1 & \text{if } n = 0;\\ a_0 b_n - a_0 b_n = 0 & \text{if } n > 0.\end{cases}\end{aligned}$$

So, $A(z)B(z) = 1z^0 + \sum_{n=1}^{\infty} 0z^n = I(z)$.

($\Longrightarrow$) Assume $a_0 = 0$. Let $B(z) = \sum_{n=0}^{\infty} b_n z^n$ be arbitrary. Then

$$[z^0](A(z)B(z)) = \sum_{k=0}^{0} a_k b_{0-k} = a_0 b_0 = 0 \neq [z^0]I(z).$$

Therefore, exists no $B(z)$ s.t. $A(z)B(z) = I(z)$. □

Since $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, the multiplicative inverse of $1z^0 - 1z^1 + \sum_{n=2}^{\infty} 0z^n$ is $\sum_{n=0}^{\infty} 1z^n$.

Lemma 9.6. *If $A(z) = 1 - z$, then the multiplicative inverse of $A(z)$ is $B(z) = sum_{n=0}^{\infty} z^n$.*

Proof. $[z^0]A(z)B(z) = a_0 b_0 = 1$, and for $n \geq 1$:

$$[z^n]A(z)B(z) = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} a_k = 1 - 1 + 0 + 0 + \ldots = 0.$$

□

$\left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2}$, which is multiplicative inverse of $(1-z)(1-z)$, and is the formal derivative of multiplicative inverse of $A(z) = 1 - z$.

Note. Manipulations of formal power series coincide with manipulations of Taylor series as long as the series converges in a neighbourhood of 0.

Proof of $(1-z)^{-1} = \sum_{n=0}^{\infty} z^n$ mult. inverse. as formal power series: We know for $|z| < 1$ that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$. So, $(1-z)\sum_{n=0}^{\infty} z^n = (1-1)\frac{1}{(1-z)} = 1$ for $|z| < 1$ implying $(1-z)(\sum_{n=0}^{\infty} z^n) = I(z)$ and we are done.

When $\sum_{n=0}^{\infty} a_n z^n$ converges on some nhood of 0, then there exists a function $A(z)$ with $\{z \mid |z| < \epsilon\} \to \mathbb{C}$ such that $\sum_{n=0}^{\infty} a_n z^n$ is a Taylor series of $A(z)$.

9.1. **Combinatorial classes.** A *combinatorial class* consists of

- a finite or countably infinite set of objects, and
- a size function defined on these objects, such that
  - the size of every object is a non-negative integer, and
  - the number of elements of any given size is finite.

The *counting sequence* of a combinatorial class is the sequence $(a_n)_{n\in\mathbb{N}}$, where a_n is the number of elements of size n in this class.

Example 9.7. Set of ordered rooted binary trees. $|T|$ = size of $T :=$ number of vertices in T. Every T has finitely many vertices for every n there are finitely many trees on n vertices.

Example 9.8. Words over the alphabet $\{a, b\}$, $|w|$ = length of w. Note: words over $\{a, b\}$ with $|w|$ the number of 'a's in w is *not* a combinatorial class, since ε, $b, bb, bbb, \ldots$ all size 0 (contradicting number of elements of any given size is finite). Counting sequence is 2^n for words over alphabet $\{a, b\}$.

Example 9.9. Ordered binary rooted trees: can count each separately, giving $1 + z + z^2 + z^2 + z^3 + \ldots$ which is equal to $1 + z + 2z^2 + 5z^3 + \ldots$.

In general: A combinatorial class size function $|\cdot|$ then the generating function of the counting sequence is

$$A(z) = \sum_{A\in\mathscr{A}} z^{|A|} = \sum_{n=0}^{\infty} \sum_{A\in\mathscr{A}:|A|=n} z^{|A|} = \sum_{n=0}^{\infty} a_n z^n.$$

We call $\sum_{A\in\mathscr{A}} z^{|A|}$ the *combinatorial form* of $A(z)$.

Let $\mathscr{A}$ and $\mathscr{B}$ be combinatorial classes. Size functions $|\cdot|_{\mathscr{A}}$ and $|\cdot|_{\mathscr{B}}$. Then

- $\mathscr{A} \uplus \mathscr{B}$ is disjoint union forming a combinatorial class with size function $|A| := |A|_{\mathscr{A}}$ for $A \in \mathscr{A}$ and $|B| := |B|_{\mathscr{B}}$ for $B \in \mathscr{B}$. For example, $\mathscr{W} = \{\varepsilon\} \uplus \{\text{non-empty words}\}$ for alphabet over $\{a, b\}$.
- $\mathscr{A} \times \mathscr{B}$ is a combinatorial class with size function $|(A, B)| = |A|_{\mathscr{A}} + |B|_{\mathscr{B}}$. For example,

$$\mathscr{W} = \{\varepsilon\} \uplus \{aw' \mid w' \in \mathscr{W}\} \uplus \{bw' \mid w' \in \mathscr{W}\} = \{\varepsilon\} \uplus (\{a\} \times \mathscr{W}) \uplus (\{b\} \times \mathscr{W}).$$

- Let $\mathrm{SEQ}(\mathscr{A})$ denote the set of finite sequences whose elements are in $\mathscr{A}$. This is a combinatorial class with size function $|(A_1, \ldots, A_k)| := \sum_{i=1}^{k} |A_k|_{\mathscr{A}}$.
- Classes $\mathscr{A}$ and $\mathscr{B}$ are *combinatorially isomorphic* ($\mathscr{A} \cong \mathscr{B}$) if there exists a size-preserving bijection between them.
- $\mathcal{E}$ is the class with 1 object of size zero, no other objects. $\mathcal{Z}$ is the class with 1 object of size 1 and no other objects. Hence,

$$\mathscr{W} \cong \mathcal{E} \uplus (\mathcal{Z} \times \mathscr{W}) \uplus (\mathcal{Z} \times \mathscr{W}).$$

We have $\mathrm{SEQ}(\mathscr{W}) = \{(), (\varepsilon), (a), (b), (ab), (aba), \ldots, (\varepsilon, \varepsilon), (a, b), \ldots, (\varepsilon, \varepsilon, \varepsilon), \ldots\}$. Infinitely many elements of size zero: (ε), $(\varepsilon, \varepsilon), \ldots$, so only a combinatorial class if $a_0 = 0$ (i.e., no elements of size 0).

Theorem 9.10. *Let $\mathscr{A}$ and $\mathscr{B}$ be combinatorial classes with generating function $A(z)$ and $B(z)$ respectively. Then*

1. *The generating function of $\mathscr{A} \uplus \mathscr{B}$ is $A(z) + B(z)$.*
2. *The generating function of $\mathscr{A} \times \mathscr{B}$ is $A(z)B(z)$.*
3. *If $a_0 = 0$, then the generating function of $SEQ(\mathscr{A}) = \frac{1}{1-A(z)}$.*

Proof. Let $\mathscr{C} = \mathscr{A} \uplus \mathscr{B}$, $C(z)$ the corresponding generating function

$$C(z) = \sum_{C\in\mathscr{C}} z^{|C|_{\mathscr{C}}} = \sum_{C\in\mathscr{A}} z^{|C|_{\mathscr{A}}} + \sum_{C\in\mathscr{B}} z^{|C|_{\mathscr{B}}} = A(z) + B(z),$$

proving (1).

Now, let $\mathscr{C} = \mathscr{A} \times \mathscr{B}$, $C(z)$ the corresponding generating function

$$C(z) = \sum_{(A,B)\in\mathscr{C}} z^{|(A,B)|_{\mathscr{C}}} = \sum_{(A,B)\in\mathscr{C}} z^{|A|_{\mathscr{A}}} \cdot z^{|B|_{\mathscr{B}}} = \left(\sum_{A\in\mathscr{A}} z^{|A|_{\mathscr{A}}}\right)\left(\sum_{B\in\mathscr{B}} z^{|B|_{\mathscr{B}}}\right) = A(z)B(z),$$

proving (2).

Let $\mathscr{C} = \mathrm{SEQ}(\mathscr{A})$ and $C(z)$ the GF. Then

$$\mathscr{C} = \biguplus_{n\in\mathbb{N}_0} \{\text{seq length } n\} \cong \mathcal{E} \uplus \mathscr{A} \uplus (\mathscr{A} \times \mathscr{A}) \uplus \ldots,$$

such that $C(z) = 1 + A(z) + A(z)^2 + A(z)^3 + \ldots$. If $A(z)$ defines a function $\{z \mid |z| < \epsilon\} \to \mathbb{C}$ for some $\epsilon > 0$, then there exists $\delta > 0$ such that $A(z) < 1$ for some $|z| < \delta$. Hence,

$$\sum_{n=0}^{\infty} (A(z))^n = \frac{1}{1 - A(z)}$$

for $|z| < \delta$, implying coefficients of Taylor series of $\sum_{n=0}^{\infty} A(z)^n$ and $\frac{1}{1-A(z)}$ coincide. □

Example 9.11. Ordered binary rooted trees, where size of T is number of vertices of T. Let $\mathcal{T}$ denote this class. Then

$$\mathcal{T} = \{\text{empty tree}\} \uplus \{\text{non-empty trees}\},$$

and we let $\overline{\mathcal{T}}$ be the set of non-empty trees. Claim: $\overline{\mathcal{T}} = \mathcal{Z} \times \mathcal{T} \times \mathcal{T}$. Size-preserving bijection is given by $T \mapsto (r, T_L, T_R)$ where r is root, T_L is subtree of left child and T_R is subtree of right child. Obviously a bijection, and size preserving since $|T| = 1 + |T_L| + |T_R|$. Thus, $\mathcal{T} \cong \varepsilon \uplus (\mathcal{Z} \times \mathcal{T} \times \mathcal{T})$. Generating function: $T(z) = 1 + zT(z)^2$ yielding $T(z) = \frac{1-\sqrt{1-4z}}{2z}$ so $[z^n]T(z) = \frac{1}{n+1}\binom{2n}{n}$.

Example 9.12. Ordered rooted trees:

- rooted trees
- order of children of a vertex matter (i.e., of T_i and T_j are subtrees, of child i and j, then swapping subtrees yields different tree unless $T_i = T_j$.
- "position" does not matter (i.e., if there is one child, don't care whether it is first, second, third, etc.)

This is a combinatorial class with size function $|T|$ number of vertices. For technical reasons: only consider non-empty trees. Question; How many ordered rooted trees on n vertices are there? Let $\mathcal{T}$ be class of ordered rooted trees. Claim: $\mathcal{T} \cong \mathcal{Z} \times \mathrm{SEQ}(\mathcal{T})$, which we note only makes sense if we do not consider the empty tree.

Size preserving bijection: $T \mapsto (r, (T_1, T_2, \ldots, T_k))$ where $k \geq 0$ is the number of children of the root. Obviously bijective, and size preserving since $|T| = 1 + \sum_{i=1}^{k} |T_i| = |(r, (T_1, \ldots, T_k))|$. Generating functions: $T(z) = z\frac{1}{1-T(z)}$ giving us $T(z) = \frac{1\pm\sqrt{1-4z}}{2}$. Since we must have $T(0) = 0$, follows that $T(z) = \frac{1-\sqrt{1-4z}}{2}$. Hence, $t_n = \frac{1}{n}\binom{2n-2}{n-1}$ for $n \geq 1$ and $t_0 = 0$, since it is just a shift of the binary rooted tree case.

Example 9.13. We have: 4 10 cent coins, 1 50 cent coins, unlimited supply of 20 cent and 1 dollar coins. How many ways to make change for n dollars from this? $\mathscr{C}$ is the set of collections of coins from our supply. $|C|$ is the value of coins in C in multiple of 10 cents. For example, $C = \{20c \times 7, 10c, 1\$ \times 2\}$, yielding $|C| = 35$. Claim: $\mathscr{C} \cong \mathscr{C}_{10} \times \mathscr{C}_{20} \times \mathscr{C}_{50} \times \mathscr{C}_{100}$, where $\mathscr{C}_i$ is the subcollection using only i cent coins. This is a combinatorial isomorphism by splitting collection into different value coins.

Now, $C(z) = C_{10}(z)C_{20}(z)C_{50}(z)C_{100}(z)$, implying

$$\begin{aligned} C(z) &= \left(1 + z + \ldots + z^4\right)\left(1 + z^2 + z^4 + \ldots\right)\left(1 + z^5\right)\left(1 + z^{10} + z^{20} + \ldots\right) \\ &= \left(1 + z + \ldots + z^4\right)\left(1 + z^5\right)\frac{1}{1-z^2}\frac{1}{1-z^{10}} \end{aligned}$$

and so

$$C(z) = \frac{1}{(1+z)(1-z)^2} = \frac{1}{2(1-z)^2} + \frac{1}{4(1-z)} + \frac{1}{4(1+z)}.$$

Theorem 9.14. *Let $\sum_{n=0}^{\infty} a_n z^n$ be power series. Exists $R \in \mathbb{R}$ such that for every $z \in B(0, R)$ converges and diverges if $|z| > R$, where R called radius of convergence of power series and $R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$.*

If f is a function and $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is a Taylor series of f at z_0, then R is the "minimal distance" of a singularity of f to z_0. That is, $\min\{|z - z_0| \mid z \text{ is a singularity of } f\}$ where f is not "well-defined plus differentiable at z".

Example 9.15. P, Q polynomials. P, W have no singularities. $\frac{P}{Q}$ has singularities when $Q(z) = 0$. $\sqrt{P(z)}$ has singularities if $P(z)$ takes positive and negative valies in a neighbourhood of z. Has 0 has singularity.

For example, $\frac{1}{1-2z+z^2}$ singularities are $z = 1$; $\frac{1}{1+z^2}$ singularities at $z = \pm i$, $\sqrt{2-z}$ singularities at $z = 2$.

Let R be the minimum absolute value of a singularity generating function of f_n. Then $\limsup_{n\to\infty} \sqrt[n]{f_n} = \frac{1}{R}$ or in other words $f_n \leq \left(\frac{1}{R}+\epsilon\right)^n$ for large n. Rule of thumb:

- "location" of singularity corresponds to exponential growth rate
- "type" of singularity corresponds to subexponential factors

Example 9.16. Domino tiling of a grid as subset of $\mathbb{Z}^2$; covering this grid by "2×1"-tile. Let $\mathcal{I}$ be the the class of domino tilings of $3\times n$ grids with weight of a tiling given by number of dominoes. Let $\mathscr{U}$ be the class of tilings which we add vertical domino in bottom right; $\mathscr{D}$ be class tiling which we add vertical domino in upper right corner. Then an element of $\mathcal{I}$ is either $\mathcal{Z}\times\mathscr{D}\uplus\mathcal{Z}\times\mathscr{U}\uplus\mathcal{Z}\times\mathcal{I}$. By reflection: $\mathscr{D}\cong\mathscr{U}$. An element of $\mathscr{D}$ is either $\mathcal{Z}\times\mathcal{I}\uplus\mathcal{Z}\times\mathscr{D}$.

$I(Z) = 2zD(z) + z^3 I(z)$ and $D(z) = z^2 I(z) + z^3 D(z)$ so $I(z) = \frac{z^3-1}{z^6-4z^3+1} = 1+3z+11z^6+41z^9+153z^{12}+\ldots$. Note $z^6-4z^3+1=0$ substitute $u=z^3$ yields $u^2-4u+1=0$ so $u=2\pm\sqrt{3}$. So, $z=\sqrt[3]{2-\sqrt{3}}$ is a singularity with minimal absolute value. Therefore, $[z^n]I(z) \leq \left(\frac{1}{\sqrt[3]{2-\sqrt{3}}}+\epsilon\right)^n$ for large n.

Example 9.17. Ordered rooted trees. $T(z) = \frac{1-\sqrt{1-4z}}{2}$. What is the exponential growth rate of t_n? $T(z)$ has a singularity at $z=\frac{1}{4}$. Hence $R=\frac{1}{4}$, and so $\frac{1}{R} = 4 = \limsup_{n\to\infty}\sqrt[n]{t_n}$. That is, $t_n \leq (4+\epsilon)^n$ for large n. In fact, $\lim_{n\to\infty} 4$ (in particular, the limit exists). We know $t_n = \frac{1}{n}\binom{2n-2}{n-1}$. Could get better asymptotics using stirling approximation. Note $\lim_{z\to 0}\frac{1-\sqrt{1-2z}}{2z} = 1$.

Example 9.18. Let $\mathcal{W}$ be the class of words over $\{a,b\}$ with no two consecutive "a"s . $\mathcal{W}\cong\mathcal{E}\uplus\mathcal{Z}^2\times\mathcal{W}\uplus\mathcal{Z}\times\mathcal{W}$. $W(z) = 1+\left(z+z^2\right)W(z)$, and so $W(z) = \frac{1}{1-z-z^2}$. Singularity if $1-z-z^2=0$, so $z=\frac{-1\pm\sqrt{5}}{2}$. Minimal absolute value is obtained by $z=\frac{-1+\sqrt{5}}{2}$. Now, $\limsup_{n\to\infty}\sqrt[n]{w_n} = \frac{2}{\sqrt{5}-1} = \frac{1+\sqrt{5}}{2}$.

9.2. **Lagrange inversion.** Call a function *analytic at* 0 if it can be represented by a Taylor series with positive radius of convergence of 0.

Theorem 9.19. (Inverse function theorem for analytic functions, special case). *Assume that f is analytic at 0, and that $f(0)$, $f'(0)=0$. Then f has an inverse function in some neighbourhood of 0, and this inverse function is analytic at 0.*

Theorem 9.20. (Cauchy's differentiation formula). *Let f be analytic at 0. Then $f^{(n)}(0) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{z^{n+1}}dz$, where γ is a simply closed curve.*

Corollary 9.21. (Lagrange inversion formula). *Assume that the function ϕ is analytic at 0 with $\phi(0)\neq 0$, and $A(z)$ satisfies the equation $A(z) = z\phi(A(z))$. Then A is analytic at 0, and $[z^n]A(z) = \frac{1}{n}[u^{n-1}]\phi(u)^n$.*

Proof. $\frac{A(z)}{\phi(A(z))} = z$, so A is the inverse function of $f : x \mapsto \frac{x}{\phi(x)}$. f is analytic at 0 because x, $\phi(x)$ are analytic and $\phi(0)\neq 0$. So, $f(0) = \frac{0}{\phi(0)} = 0$. $f'(x) = \frac{1}{\phi(x)} - \frac{x\phi'(x)}{\phi(x)^2}$ implies $f'(0)\neq 0$. By inverse function theorem, A is analytic at 0.

Observe

$$\begin{aligned} n[z^n]A(z) &= [z^{n-1}]A'(z) \\ &= \frac{(A')^{(n-1)}(0)}{(n-1)!} \\ &= \frac{n}{2\pi i}\oint_\gamma \frac{A'(z)}{z^n}dz \\ &= \frac{n}{2\pi i}\oint_\gamma \frac{1}{\left(\frac{u}{\phi(u)}\right)^n} \\ &= \frac{n}{2\pi i}\oint_\gamma \frac{\phi(u)^n}{u^n}du \\ &= [u^{n-1}]\phi(u)^n. \end{aligned}$$

□

$u = A(z)$, $\frac{du}{dz} = A'(z)$, $z = \frac{A(z)}{\phi(A(z))} = \frac{u}{\phi(u)}$.

Example 9.22. Ordered rooted trees. $\mathcal{T} \cong \mathcal{Z} \times \mathrm{SEQ}(\mathcal{T})$ so $T(z) = \frac{z}{1-T(z)}$. Hence, $[z^n]T(z) = \frac{1}{n}[u^{n-1}]\frac{1}{(1-u)^n} = \frac{1}{n}\binom{2n-2}{n-1}$. For

$$\begin{aligned} \left(\frac{1}{(1-u)^n}\right)' &= \frac{n}{(1-u)^{n+1}} \\ \left(\frac{1}{(1-u)^n}\right)'' &= \frac{n(n+1)}{(1-u)^{n+2}} \\ \left(\frac{1}{(1-u)^n}\right)^{(n-1)} &= \frac{n(n+1)\dots(2n-2)}{(1-u)^{2n-1}}. \end{aligned}$$

Hence,

$$[u^{n-1}]\,\frac{1}{(1-u)^n} = \left(\frac{1}{(1-u)^n}\right)^{(n-1)}\Bigg|_{n=0}\frac{1}{(n-1)!} = \binom{2n-2}{n-1}.$$

Example 9.23. Ordered rooted binary trees. $\mathcal{T} \cong \mathcal{E} \uplus (\mathcal{Z} \times \mathcal{T} \times \mathcal{T})$. $T(z) = 1 + zT(z)^2$. Consider $F(z) = T(z) - 1$. $F(z) + 1 = 1 + z(F(z)+1)^2$, so $F(z) = z(F(z)+1)^2$. $[z^n]T(z) = [z^n]F(z)$ for $n \geq 1$. Hence,

$$[z^n]F(z) = \frac{1}{n}[u^{n-1}](u+1)^{2n} = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}.$$

9.3. **Cayley's Theorem.** Recall Cayley's Theorem, which states there are n^{n-2} spanning trees of K_n. We will prove there are n^{n-1} rooted spanning trees (this is enough, there are n ways of choosing a root). Let t_n be the number of rooted spanning trees of K_n. Let $r_n := \frac{t_n}{n!}$, $R(z) = \sum_{n=0}^{\infty} r_n z^n$. $\sum_{n=0}^{\infty} t_n z^n = \sum_{n=0}^{\infty} n^{n-1} z^n$ does not converge for $z \neq 0$. This is the reason to consider r_n instead. The power series $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is called the *exponential generating function of* a_n.

Rooted spanning tree of K_n:

(1) pick a root
(2) choose degree of the root
(3) choose the size of the subtree for each neighbour of the root
(4) choose vertices of each such subtree

Hence,

$$t_n = n\sum_{k=0}^{n-1}\left(\sum_{j_1,\dots,j_k\geq 1, j_1+\dots+j_k=n-1}\frac{(n-1)!}{j_1!\dots j_k!}\prod_{i=1}^{k} t_{ji}\right)\frac{1}{k!}.$$

So,

$$r_n = \sum_{k=0}^{n-1}\frac{1}{k!}\sum_{j_1,\dots,j_k\geq 1, j_1+\dots+j_k=n-1}\prod_{i=1}^{k} r_{ji} = \sum_{k=0}^{\infty}\frac{1}{k}[z^{n-1}]R(z)^k = [z^n]R(z).$$

So,

$$[z^n]R(z) = [z^n]z\sum_{k=0}^{\infty}\frac{1}{k!}R(z)^k \implies R(z) = ze^{R(z)}.$$

Apply Lagrange inversion:

$$\begin{aligned}
[z^n]R(z) &= \frac{1}{n}[u^{n-1}](e^u)^n \\
[z^n]e^{\alpha z} &= \frac{\alpha^n}{n!} \\
[z^n]R(z) &= \frac{1}{n}\cdot\frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!} \\
\implies \qquad t_n &= n^{n-1}.
\end{aligned}$$

Gamma function: $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$, $\Re(z) > 0$.

Note. This also works for asymptotics of any generating function of the form $(1-z)^\alpha$ - can also be used for much more precise asymptotics.

Fact: $\frac{1}{\Gamma(z)} = -\frac{i}{2\pi}\int_\gamma (-t)^{-z}e^{-t}dt$ and $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$.

$T(z) = \frac{1-\sqrt{1-4z}}{2}$. $t_n = \frac{1}{n}\binom{2n-2}{n-1}$ which by stirling approximation is $\approx \frac{4^{n-1}}{\sqrt{\pi n^3}} = 4^n\frac{1}{4\sqrt{\pi n^3}}$. $t_n = 4^n\frac{1}{2}[z^n]\left(-\sqrt{1-z}\right)$ for $n \geq 1$. Let $z := 1 + \frac{t}{n}$ so $\frac{dz}{dt} = \frac{1}{n}$. Hence,

$$\begin{aligned}
[z^n]\left(-\sqrt{1-z}\right) &= -\frac{1}{2\pi i}\oint_\gamma \frac{\sqrt{1-z}}{z^{n+1}}dz \\
&= \frac{i}{2\pi}\int_{\tilde{\gamma}}\frac{\sqrt{-t/n}}{(1+t/n)^{n+1}}\frac{1}{n}dt \\
&= \frac{i}{2\pi}\frac{1}{\sqrt{n^3}}\oint_{\tilde{\gamma}}(-t)^{\frac{1}{2}}\left(1+\frac{t}{n}\right)^{-n-1}dt \\
&\approx \frac{i}{2\pi}\frac{1}{\sqrt{n^3}}\oint_{\tilde{\gamma}}(-t)^{\frac{1}{2}}e^{-t}dt \\
&= \frac{1}{\sqrt{n^3}}\left(-\frac{1}{\Gamma\left(-\frac{1}{2}\right)}\right) \\
&= \frac{1}{2\sqrt{\pi n^3}}.
\end{aligned}$$

Thus, $t_n \approx \frac{4^n}{4\sqrt{\pi n^3}}$.